Honors Coordinate Geometry
(Draft)

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Students who desire to learn the subject of Coordinate Geometry need to study all of the sections up to this point. This is the dividing line between regular and advanced Coordinate Geometry. Students who continue further will be challenged at higher levels and will gain experience with more advanced mathematical problem solving concepts and techniques. In any case check out Challenging Problems.

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Preface

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Important: Before doing any problems in this book, read those parts of the Preface which are underlined and highlighted in red.

Students should successfully complete Algebra I and a course in a proof based Classical Geometry before taking this course in Coordinate Geometry. The book's purpose is to give all who desire, further opportunity to become able mathematical problem solvers. This book does a thorough job of covering the subject of Coordinate Geometry, however its prime reason for being, its focus, is to teach mathematical problem solving, Math education has declined in American secondary schools since the late 1980's so the opportunity to learn mathematical problem solving well, isn't always available anymore.

This book is quite suitable to be used in a teacher classroom situation. When this book is combined with its answer book it is suitable for self study. (Answers are at the end of this book, answer book is still in the process of being completed). Perhaps the best possible way to study this material is with the aid of a knowledgeable tutor. In a junior high or high school setting, typically one school year is allowed for the study of Geometry. Given that Classical Geometry would take up most of the school year, only parts of this book should be covered. Recommended would be "Proving Theorems using Coordinate Geometry" and those parts leading to it.

Students studying this book should attempt to work out all problems, proofs and derivations on their own before looking at the answers. If it is necessary to look at a solution after you have struggled unsuccessfully, look at it only briefly, just long enough to get a needed clue, then continue to work out the problem on your own. The more you do a problem on your own, by trying and perhaps failing, pondering and struggling, the more benefit you will get by doing the problem. Once you have completed a problem, look at the provided problem solution in the answer section, in many cases you will get additional perspective on problem solving by looking at other solutions.

Coordinate Geometry and Classical Geometry are different in that Coordinate Geometry makes use of a coordinate system (usually the Cartesian coordinate system). Coordinate Geometry also makes more extensive use of algebra. Coordinate Geometry proofs are generally more straight forward than those of Classical Geometry, and generally require less ingenuity when solving the same problem. More often than in Classical Geometry the struggle of Coordinate Geometry is not trying to find how to solve a problem, but how to solve it elegantly. Coordinate Geometry proofs more often than Classical Geometry Proofs can become very complex and voluminous if care is not taken. See Napoleon's theorem (in the trigonometry book, the next book in this series) and the theorem in this book that states that the three medians of any triangle meet at a common point for examples of proofs that are particularly elegant). In this book, using Coordinate Geometry, we also prove Napoleon's theorem, this Coordinate Geometry proof is not as elegant (it is more cumbersome) as the Trigonometry proof. Analytic Geometry is a form of Coordinate Geometry. Coordinate Geometry deals with problems and figures such as lines, triangles, squares, etc. that are dealt with in Classical Geometry, while Analytic Geometry deals with figures that are unique to itself, such as parabolas, ellipses, hyperbolas and cycloids etc. Circles are dealt with in Coordinate Geometry and Analytic Geometry. The boundary between these two Geometrys is a fuzzy one. The technique Locus is used in coordinate geometry to find the equation of a circle, but other than this, is mostly used only in analytic geometry. This study of Coordinate Geometry, goes a little beyond the boundaries of what is considered Coordinate Geometry. Several locus problems are included as are some other topics** in order to provide the student with a more full mathematical problem solving experience.

In Trigonometry additional methods will be developed that give increased ability to deal with angles. Analytic Geometry makes use of these methods.

*In locus problems, a geometrical definition of a graph is given and the task is to find the equation of the graph.

**In addition to giving students the opportunity to solve several interesting locus problems, this book teaches students how to calculate slopes of certain curves using a non calculus Coordinate Geometry technique, and then uses this ability to solve various interesting problems. Vectors and parametric equations of lines are also introduced and used minimally in this book, as are the techniques of making simplifying assumptions (size of earth problem) and postulate introduction (slope of curved graphs section). Simplifying assumptions and postulate introduction are often used in science and engineering. These techniques allow some problems to be solved which otherwise could not be, or allow problems that are otherwise extremely difficult to solve, to be solved with greater ease. As people we postulate and make simplifying assumptions all the time in daily life without thinking about it. In math and science, for most people, these techniques need to be taught.

When any problem in this book gives any parameter in decimal numbers, the student's answer should be in decimal form. Ideally when all problem parameters are given in integer form, answers should be given in integer form and not contain decimal numbers.

However in some problems doing the calculations in integer form would be overly complex. If when solving a problem using integers the calculations become overly complex due to the fact that decimal numbers are being avoided, it is recommended that the student do the problem, using decimals, (4 or more digits should be used). This will allow the student to focus on solving the problem at hand and not be weighed down by the complexity of the calculations.

Section 1 should be covered first, before anything else in this book. Section 1 covers equations of lines and non-rotated parabolas; how to find the intersection between lines and lines, lines and parabolas, parabolas and parabolas; how to find the roots of 2nd order polynomials; algebra proofs; shifting of functions; how to find the non rotated parabola which passes through any three non collinear points; and other interesting concepts which are not made apparent in most other courses of algebra. Doing Section 1 will help ensure there is an adequate Algebraic foundation for Coordinate Geometry foundation and will significantly improve a student's Algebra skills.

Some proofs of problems in the Challenging Problems section use vector techniques. If students want to understand these proofs the student will need to study appendix 1. Appendix 1 covers vector addition, the proportional point formula from a vector perspective, vector rotation by +/- 90 degrees at a time and some other vector topics. Sometimes using vector techniques in proofs simplifies the notation and can make proofs easier to see and to understand the underlying logic. It is the goal in most of this book to not deviate from the spirit of classical coordinate geometry too far. Using vectors in some proofs increases the ability of the student to solve some hard to do problems because of the increased clarity and simplicity that vector notation provides. (The proof of the theorem in the Challenging Problems section that the 3 medians of a triangle all meet at one point is an excellent example of this).

Viewing Answers

Sections in this book which have problems are numbered. Problems that have answers available in the answer section will be numbered with the section number, followed by a decimal point, followed a the problem number. For example, if a problem is numbered as 7.4, or 4.2.1 (this number format indicates this problem has an answer available in the answer section). Do a search on 7.4) or 4.2.1) to find the solution, and again to return to the problem.

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1 Algebra Review
Quadratic Equations

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This chapter is meant to review parts of algebra that will be of use in the study of coordinate geometry. Parts of this section may be a review of what the student learned in Algebra I. However those parts were not left out for two reasons. 1. for completeness sake, i.e. this book strives to be self contained on the subject of coordinate geometry and key supporting subjects. 2. Review is good.

This section familiarizes students with polynomials and teaches how to solve quadratic equations, i.e. given the equation $Ax+By+C=0$, for what value(s) of x is the equation $3x^2-10x+6=0$ true? $2x^2-12x+25=0$ is an example of a quadratic equation with numeric coefficients. It is necessary to solve quadratic equation in order to solve some coordinate geometry problems. Therefore the student needs to know how to solve quadratic equations, and how the ability to do this was derived in the first place. It is the philosophy of this book that whenever the student is led to a mathematical fact or a method of how to solve a problem, that the student should also be led to an understanding as to why the fact is true, or why method works, and if possible how to derive this fact for themselves. This involves careful explanation, presenting proofs of theorems and derivations of formulas, and challenging the student to derive formulas and come up with their own proofs. Experience with proofs and derivations are key components enabling discovery in mathematics, physics and engineering. If we want a scientific professionals who are highly capable and will lead in their fields, we can't spoon feed them facts and methods in their formative years, i.e. in junior high and high school.

Polynomials and Their Roots

Polynomials can be expressed in two forms

Sum of Product (SOP)	<->	Product of Sums (POS)
$4x^2+16x+12$	<->	$4(x+1)(x+3)$

The above polynomials are equal and are each 2nd order polynomials. The student should multiply the terms on the right side, $4(x+1)(x+3)$ together and verify they equal the terms on right side, $4x^2+16x+12$. A polynomial can be any order N where N = {1,2,3 ...} is a counting number. The expressions $4x^2+16x+12$ and $4(x+1)(x+3)$ were referred to as polynomials, but in reality they are one polynomial, they are merely different forms of the same polynomial. Examples of polynomials expressed in both forms are shown below.

```
            Sum of Product <-> Product of Sum
1st order:           2x+2 <-> 2(x+1)
2nd order:     5x^2+20x+15 <-> 5(x+1)(x+3)
3rd order:   x^3+2x^2-5x-6 <-> (x+1)(x-2)(x+3)
  .
  .
  .
Nth order:

SOP form   A0 + A1*x + A2*x^2 + A3*x^3 ... An*x^n
                     <->
POS form   An(x1-r1)(x2-r2)(x3-r3) ... (xn-rn)
```

The student can verify that the An’s in each form are equal.

1) Verify that the SOP and POS forms of the following polynomials are equal.

```
   Sum of Product             <->    Product of Sum
a) 1st order:  2x+2           <->    2(x+1)
b) 2nd order:  5x^2+20x+15    <->    5(x+1)(x+3)
c) 3rd order:  x^3+2x^2-5x-6  <->    (x+1)(x-2)(x+3)
```

Notice that in each of the polynomials expressed above, the order of the polynomial can be determined from a polynomial in SOP form by looking at the term with the highest exponent of x. For example in a second order polynomial, the term with the highest exponent of x is 2, likewise in an 8th order polynomial the term with the highest exponent of x would be 8. The following is an 8th order polynomial, x^8+1. The fact that the terms involving x^7, x^6, or the rest of the terms are missing doesn’t matter. In reality these terms aren’t missing, their coefficient are zero.

When a polynomial is expressed in POS form i.e. (x-1)(x+2), the order of the polynomial can be determined by now many monomial products there are, (x-3) is a monomial. Notice above that a 2nd order polynomial in POS form has two monomials multiplied by each other, a third order polynomial in POS form has 3 monomials multiplied by each other. A 1st order polynomial in POS form has one monomial, an N order monomial in POS form would have N monomials multiplied by each other. If any polynomial is of a certain order when expressed in one form, it will also be of that same certain order when it is expressed in the other form.

2) Give the order of each of the following polynomials and state if it is in POS or SOP form.
   a) (x-3); b) (x-3)(x+7); c) 3x^3+7x+1; d)x^7+x^4+3x^2+1;
   e) 16(x-8)(2x-7)(3-x)(7-1)(3-7x); 1+17x^5+11x^32 ... +5x^n.

Roots of Polynomials

A very useful thing to do with a polynomial is to find its roots, or those values of its variable (we use the variable x) which make the polynomial equal to zero. For example the polynomial below

5x^2+20x+15 <-> 5(x-2)(x+3)

has roots of 2 and –3. This is easy to see by substituting either 2 or –3 into 5(x-2)(x+3). Substituting –2 into (x+2) will make this term zero. Likewise substituting –3 into (x+3) will make that term zero. Whenever any term of a polynomial in POS form is zero, the value of the entire polynomial is zero. Whenever a polynomial is in POS form, finding the zeros of the polynomial is easy. It is more difficult to find zeros of a polynomial when it is in SOP form.

3) What are the roots of each of the following polynomials?
   a) (x-1)(x+2); b) 12(x-5)(x+6)(x-11);

4) Find the SOP form of each of the following polynomials, then verify that the SOP form of each polynomial has the same set of zeros as its POS form. a) (x+5)(x+2); b) (x-1)(x+2);
   c) (x-1)(x-1)(x+5).

Fact: If two polynomials have same zeros as each other, they are equivalent polynomials.

Changing Forms of a Polynomial SOP <-> POS

To convert a polynomial from POS form to SOP form is not difficult, one must multiply the monomials and leading coefficient together, gather the like terms together and then arrange them in the correct order. In order to convert a polynomial from SOP form to POS form, one must find the roots of the polynomial r1 .. rn, along with the leading coeficient An and then put these in the following form

An(x-r1)(x-r2)(x-r3) ... (x-rn)

where r1, r2, r3 .. rn are the roots of the polynomial.

Considering a polynomial in POS form i.e. (x-1)(x+2)(x-3), it is easy to see that an nth order polynomial has n roots. Any n order polynomial in SOP form also has n roots. This is because when it is converted to SOP form, it will continue to be an nth order polynomial and will have the same roots as before. The fundamental theorem of Algebra tells us that any nth order polynomial has exactly n roots, making it easy to show that any order n polynomial of one form can be converted to an nth order polynomial of the other form. It is somewhat obvious that any polynomial in POS form can be converted to SOP form. It is not as obvious that all polynomials of SOP can be converted to POS form. It takes the fundamental theorem of algebra to make this evident.

5) What does the Fundamental Theorem of Algebra say?

6) a) Given the Fundamental Theorem of Algebra, assuming you have the ability to find the roots of any polynomial, explain how any polynomial of SOP form can be converted to POS form. b) Also explain how any polynomial in SOP form could be converted to POS form.

The 2nd polynomial (x-1)(x-1) also has 2 roots, its roots are 1 and 1. This polynomial is said to have a double root of 1.

7) How many roots do each the following polynomials have.
a) x^2-4x+2; b)(x-1)(x+2); c) x^7-x^3+12=0;
d)(x-1)(x-1)(x-1)(x+3); e) x^9-26=0.

While finding a root of a polynomial in POS form is easy, finding the roots of a polynomial in SOP form is not as easy. Here we will learn two main ways to find the roots of polynomials. The first method will work for any polynomial of any order that has only integers as its roots. The second method will work for all polynomials of 2nd order, here the roots don’t need to be integers, they can be any type, fractional i.e. 2/3, irrational i.e. 1+sqrt(2) or complex i.e. 2-sqrt(-1).

Any polynomial of second order which is set to zero is referred to as a quadratic equation. 3x^2-2x+5=0 is a quadratic equation. From here on the discussion of polynomials, will focus on polynomials of 2nd order. This is because in this book of coordinate geometry, we will work with parabolas, which are polynomials of 2nd order. In this book of coordinate geometry, we do not focus on polynomials of order 3 or higher. Though method 1 below will work for polynomials of any order where the where the roots of the polynomials are integers. The discussion of it will only deal with polynomials of order 2. With minor modification of what is learned here the student may apply this method to polynomials of higher order.

Finding Roots of Quadratic Equations (Method 1)
Consider the second order polynomial (x+a)(x+b). When this polynomial is converted to SOP form it becomes x^2+(a+b)x+ab. Notice that the last coefficient (the coefficient of x^0), ab is the product of a and b. Notice also that the middle term (a+b) is the sum of the terms a and b. Therefore if you have such a polynomial in SOP form and want to find its roots, you need to find all the factors of the last term, and then see which of those factors add up to (a+b). Put these factors a and b, into a polynomial of the form (x+a)(x+b), from here it is easy to calculate the roots of the polynomial, i.e. –a and –b.

Example 1: Find the roots of the polynomial x^2+7x+12. The factor pairs of 12 (for simplicity sake we do not write down the negative possibilities for a and b, but we always keep in mind that negatives may exist) are (1,12), (2,6),(3,4). Of these, only the 3 and the 4 add to 7. Therefore this polynomial converted to POS form is (x+3)(x+7) and the roots or zeros of this polynomial are –3 and –7.

Example 2: Lets consider an other example, x^2+4x-12. Again we write only the positive factors, but we keep in mind that negatives may exist. Therefore we list again all the factor pairs of 12 (1,12), (2,6),(3,4). Of these factor pairs, only (2,6) can add to 4, where –2+6=4. Therefore this polynomial in POS form would is (x-2)(x+4) and the roots of this polynomial are 2 and –4.

Example 3: If you need to solve a polynomial with a leading coefficient other than 1, such as 2x^2-26x+72, factor out the leading coefficient, 2(x^2-13x+36) and proceed as before. The factor pairs of 36 are (1,36),(2,18),(3,12),(4,9) and (6,6). Of these, only (4,9) add to -13, where –4-9=-13. Therefore the POS form of this polynomial is 2(x-4)(x-9), and the roots of this polynomial are 4 and 9.

8) Convert all of the following polynomials into POS form and then find the roots of these polynomials. a) x^2+3x+2; b) x^2+2x+1; c) 3x^2-3x-6; d) 3x^3-15x-108; e) a^2-35a+300.

Finding Roots of Quadratic Equations (Method 2)
Consider (x+a)(x+a)=0 and the same polynomial written in SOP form as x^2+2a*x+a^2=0. Notice that the last term, (a^2) equals half of the middle coefficient (2a) squared. Therefore any polynomial of the form x^2+2a*x+a^2 may be written as (x+a)^2 (a perfect square). It is easy to find the roots of any polynomial that can be written as a perfect square. We do this as follows x^2+2a*x+1=0 -> (x+a)^2=0 -> x+a=0 -> x=-a

Consider x^2+2x-24. It is not possible to write this polynomial as a perfect square, nevertheless we may perform a procedure called completing the square and then write this polynomial in terms of a perfect square. Once this is done it will be quite easy to find the roots of this polynomial. This is done as follows.

Completing the square example
x^2+2x-24=0 -> x^2+2x+1 –1 -24 =0 (Notice here we added a 1 to form a perfect square, but then we had to subtract a 1, so the equation will remain unchanged). -> (x+1)^2 –1 -24=0 -> (x+1)^2-25=0 -> (x+1)^2=25 -> x+1= +/- sqrt(25) -> x+1= +/- 5 -> x=-1 +/- 5 -> x=-6 or x=4. This answer is easily verified using method 1 or by substitution of either answer into the original equation.

Since –6 and 4 are the zeros of this polynomial, this polynomial converted to POS form is written as follows. (x-{-6})(x-4) or (x+6)(x-4). We note here that when taking the square root of both sides of an equation as in (x+1)^2=25 -> x+1= +/- sqrt(25), it is necessary to employ the +/- because all real numbers n have 2 square roots +sqrt(n) and –sqrt(n), and when doing this we want to capture both square roots, so that we may obtain all roots of this second order polynomial.

Method 2 may also be used with polynomials that have a coefficient other than 1 for the x^2 term. In such a case, just divide both sides of the equation by the coefficient of x^2. To solve the polynomial 2x^2+2x-60=0 using method 2. Begin by dividing both sides of the equation by 2, making the equation x^2+x-30=0. From here, continue on with the rest of the steps.

9) a) Use method 1 to find the zeros of the following polynomial. b) Check your answer by substituting the zeros into the polynomial to verify they are indeed zeros. c) Use method 2 to find the zeros of this polynomial. Verify these zeros by comparing them with the zeros you got in a. c) Use the zeros of this quadratic equation to put it into POS form. d) Once you have this quadratic equation in POS form, convert it back into SOP form by multiplying the monomials together. The result should be the same as the original equation. .. x^2+x-72=0.

16.10) a) Using method 2, find the zeros of Ax^2+Bx+C=0. Hint: If you find this difficult, do this problem first using numbers as coefficients, then use this as a guide to do the given problem. b) Most likely, the expression you derived in part a was not in the same form as the expression given below. Using algebraic manipulation, put the expression you derived in part 'a' into the following form.

```
-B +/- sqrt(B^2-4AC)
--------------------    <- this expression is referred to as
        2A                 the quadratic formula.
```

11) The above expression is referred to as the quadratic formula and this is its standard form. Memorize the quadratic formula both verbally and in writing as directed in problems 11 and 12 (do this now). Make use of the quadratic formula to find zeros of the quadratic equations that follow. Once you find the zeros of the following equations check your answer using a method of your choice. a) x^2+33x+200=0; b) 2x^2+94x-300=0; c) x^2+4x+-1=0.

12) State the quadratic formula without looking. Do this several times until you are confident you have committed it to memory. The quadratic formula is typically verbalized as follows.

"minus B, plus or minus the square root of ..
B squared minus 4 A C ..
all over 2A"

13) Write the quadratic formula without looking. Do this several times until you are confident you have committed it to memory.

14) Prove that $\frac{-B + \sqrt{B^2-4AC}}{2A}$ and $\frac{-B - \sqrt{B^2-4AC}}{2A}$ are zeros of the quadratic equation $Ax^2+Bx+C=0$. Do this by substituting each of these expressions into Ax^2+Bx+C.

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2 Cartesian Coordinate System
Lines, General Points, Shifting Theorem, Parabolas

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Important: Before doing any problems in this book, read those parts of the Preface which are underlined and highlighted in red. Also when graphing lines or other functions, it is best (by far) to use graph paper. Take the time to get some now before you begin.

This section we study the Cartesian coordinate system, lines, the shifting theorem and parabolas. Some of what we learn in this section will be a review of what was learned in Algebra I, much of This review is warranted to refresh the students understanding and for completeness sake. This book strives to present complete coverage of coordinate geometry. Unlike many math books of today, most every assertion will be proved. It is a goal to never to have students following recipes to solve problems where they don't understand why what they are doing works. Where feasible, theorems will be proved, equations will be derived and explanations will be given sufficient to provide the student with a foundational understanding. Another goal of this book is to inspire and to impart a love of problem solving and of mathematics. These two goals support each other. The best learning occurs when students learn to do something on their own by thinking, pondering and yes individual struggle. Consequently a method of study taught in this book is .. when a proof or example is given, to encourage the student to try to solve the problem or proof on their own before looking at the provided solution. We will be doing problems that the student (hopefully will consider worthwhile and inspiring. This aim is to go deep enough into the subject to give the student a sense of accomplishment and to do it in a way that will increase the students love of mathematics. For example, in this section we prove that if the graph of a non rotated parabola (a parabola whose axis is vertical) is added to the graph of a non vertical line, the resulting graph is the original parabola that has been shifted to a new location. If these two parabolas were to be laid one on top of the other, they would overlay each other perfectly. This is quite an interesting geometrical fact and proving this helps gives students experience making use of algebra to solve geometry problems.

Cartesian Coordinate System

The Cartesian coordinate system was invented by a great French philosopher and mathematician named Rene Descartes in the 1600's. Legend has it he was laying on his bed looking at a bug crawl across the ceiling and it occurred to him he could describe the path the bug was taking using a mathematical equation. With the discovery of a coordinate system, it became possible define geometric shapes and solve geometry problems in a new and different way than had been done for thousands of years sine the ancient Greeks gave us Classical geometry. When geometry is done using a coordinate system, we call this coordinate geometry. Some theorems that are difficult to prove using classical geometry are easy to prove using coordinate geometry and visa versa. The invention of the Cartesian coordinate system ranks as one of the greatest inventions of all time. You should already have had some exposure to the Cartesian coordinate system when you took Algebra I. We will review its basics here and then to on to make use of it so solve many of the same problems that the student solved when they took classical geometry.

A Cartesian coordinate system consists of a plane that contains two perpendicular number lines. The 0's of the number lines coincide, (they are at the same location). The place where these 0's coincide is called the origin. The scales of these number lines are the same, meaning that the distance between any two numbers A and B on one of the number lines is the same distance between those same two numbers on the other number line. We call one of these number lines the x axis, and the other number line the y axis. When viewing a Cartesian coordinate system, it is typically though of as being flat parallel to the ground or vertical, i.e. parallel to a wall. The x axis is 'extends from side to side' and is sometimes referred to as the horizontal axis. The y axis extends vertically, or towards and away from the person viewing the coordinate system. On the x axis, positive numbers are to the right of the origin and negative numbers to the left. On the y axis positive numbers are above the origin or are located on the part of the y axis that extends away from the person viewing the coordinate system. The negative numbers are located below the origin or are located on the part of the y axis that extends into the person viewing the coordinate system.

What we have just described is a 2d or 2 dimensional Cartesian coordinate system. A Cartesian coordinate system may also exist 3 dimensions. In such a case the x-y plane is thought of as being horizontal, and there is a z axis which is vertical. Positive numbers of the z axis are above the x-y plane, negative numbers of the z axis are below the x-y plane. The dimension of Cartesian coordinate system corresponds to the number of 'number lines' it makes use of. A single number line can be though of as a 1 dimensional Cartesian coordinate system. There are geometry books where students are taught to solve problems involving 4 (and more) dimensions.

The location of any point that exist in the space of a coordinate system can be specified by an 'address' or set of coordinates. (For a 2 dimensional coordinate system this space is the plane of the coordinate system). For example, if P is a point of a 2d Cartesian coordinate system, if the projection from P to the x axis is 2, and if the projection from P to the y axis is -3, the coordinates of P are (2,-3). 2 is the x coordinate of P and -3 is the y coordinate of P. Put another way, if a vertical line through P intersects the x axis at 2, and a horizontal line through P intersects the y axis at -3, the coordinates of P are (2,-3). The x and y axis of a 2d Cartesian coordinate system divide the the x,y plane into 4 main sections called quadrants. If the x and y coordinates of a point are positive the point lies in the 1st quadrant. If the x coordinate of a point is negative and the y coordinate is positive, the point lies in the 2nd quadrant. If both coordinates are negative, the point lies in the 3rd quadrant. If the x coordinate of a point is positive and the y coordinate is negative, the point lies in the 4th quadrant.

Exercise: a) Using a piece of graph paper if available, draw a 2d Cartesian coordinate system, label the x axis from -9 to 9 and the y axis from -9 to 9; a) Locate and mark the following points on your coordinate system: the origin, i.e. (0,0), (-4,7), (5,5), (7,-2), (-3,-2); b) Put a roman numeral I in the 1st quadrant of your coordinate system. Put a II in the 2nd quadrant. Put a III in the 3rd quadrant, put a IV in the 4th quadrant. c) Using a pencil lightly shade the 1st and the 3rd quadrants.

Slope of Lines

The student learned in Geometry that any two points specify a line. One point and an orientation (direction) also specify a line. The slope of a line gives an indication of the orientation of the line.

If a line contains the two points P1(0,0) and P2(1,2), going from P1 to P2, x (the x coordinate) changes by 1 and y (the y coordinate changes by 2. Slope of a line, m (in a 2d Cartesian coordinate system) is defined as

```
    change of y
m = -----------
    change of x
```

(we use the letter m as a symbol for slope)

Therefore the slope of this line is m = 2/1 or 2.

Reminder: (When studying this book and always when studying
(mathematics, you should try to do a problem for yourself
(before you look at the solution.

Example Problem

A line L contains the two points P1(x1,y1) and P2(x2,y2). What is the slope of the line L?

Solution

In going from P1(x1,y1) to P2(x2,y2), x changes by x2-x1 and y changes by y2-y1, therefore

$$m = \frac{\text{change of y}}{\text{change of x}} = \frac{y2-y1}{x2-x1}$$

In going from P1 to P2 we stated x changes by x2-x1. This stems from the fact that x starts out as x1 and ends up as x2.

x1 + 'change of x' = x2 -> .. (-> here means implies)

'change of x' = x2-x1

Change of y in going from P1 to P2 can be calculated similarly.

Exercise) In the previous example, in calculating the slope it was assumed you went from P1 to P2. Do this problem again by going from P2 to P1. Do you get the same slope? (the answer is yes)

It doesn't matter which 2 points of a line you make use of to calculate its slope, you will get the same answer no matter which two points of the line you use. In a later problem, students will be asked to prove this.

The definition of slope (of a line) is sometimes referred to as

$$m = \frac{\text{the rise}}{\text{the run}}$$

Where 'the rise' is change of y and 'the run' is change of x, in going from one point on the line to an other.

Slope of Lines Exercises

1) Give two ways to specify a line.

2) Draw lines with the following slopes. a) 0; b) 1; c) 2; d) 3; e) -1; f) -5; g) 2/3; h) –5/8; h) undefined.

3) Find the slope of the line that passes through each of the following point pairs and then graph the line. a) (0,0) (1,1); b) (4,-7) (2,7); c) (x1,y1) (x2,y2)

4) Prove: On a number line, when going from x1 to x2, the change of (the value of) x is x2-x1.

One thing we should mention before we begin our discussion of line equation types is, in discussions below we introduce several line equation types and then state that this is the slope intercept (for example) form of a line equation. In doing this we are implying that (in the 2d Cartesian Coordinate system), the equation types we introduce do indeed represent lines. We do not prove this and the reason we don't is because doing this isn't possible. The fact that any of these line equation types represent lines must be postulated. Our inability to prove this stems from the fact that lines are undefined objects in geometry. Perhaps you remember this from your study of classical geometry.

In the discussion below and from now on in this book, unless it is stated otherwise, the words coordinate system refers to a 2 dimensional coordinate system.

Lines: Slope Form

The simplest possible line equation is y=x. This line is made up of all the points in the space of the coordinate system where the y coordinate of the point equals the x coordinate. For example (0,0), (1,1) and (-2,-2) are all points of (the line) y=x. The points (1,2), (-3,7), (6,8) are not points of y=x. Given that two points specify a line, it is necessary to determine only two points of any line to graph it. The simplest form of line equation which we give a name to is the slope form. y=mx is a line equation in slope form. m it turns out is the slope of the line y=mx. y=2x, y=3x, y=1x, y=0x, y=-2x are examples of lines in slope form.

Exercise) a) Calculate the slope of the following lines; b) Graph the following lines on the same piece of graph paper , do this by determining at least 2 points of each line and making use of these two points to graph the line. I) y=x; II)y=2x; III)y=3x; y=-x or y=(-1)x

Did you notice when graphing these lines, the greater m is, the steeper the line is?

Lines: Slope Intercept Form

y=mx+b is the slope intercept form of a line. Did you notice that the slope form of a line always intersects the y axis at 0? i.e. at the point (0,0). The b in the slope intercept form of the line has the effect of moving the line up by the amount b. (Up by an amount of -3 means down by an amount 3). Therefore the line y=mx+b intersects the y axis at b, i.e. y=mx+b intersects the y axis at the point (0,b). Take the time to visualize why this would be the case.

Lines: Slope Intercept Form Exercises

1) Graph each of the following lines using graph paper by first determining at least 2 points of each line and then making use of these points to graph the line. a) y=3x-2; b) y=x-1; y=-x+2

2) Prove that the slope of the line y=mx+b is m.

3) Prove that the y intercept of the line y=mx+b is b, i.e. prove that the point of intersection of this line and the y axis is (0,b).

Lines: Point Slope Form

What if we want a line (equation) with a slope of 2, that contains the point (1,3). We can make use of the slope intercept form of a line to determine the line (equation) we want. As always, try to do the following exercise yourself before looking at the provided solution.

Example Exercise) Make use of the slope intercept form of a line to determine a line (equation) that includes the point (1,3) and has a slope of 2.

Since we are making use of the slope intercept form of a line and the line we desire has a slope of 2, the line we seek has the form y=2x+b, where b is yet to be determined. Since the line we seek includes the point (1,3), we substitute x=1 and y=3 into y=2x+b, this gives us (3)=2(1)+b, this implies b=3-2=1. Substituting b=1 into y=2x+b is y=2x+1, therefore the line (equation) we desire is y=2x+1. Finally we check our solution by plugging the point (1,3) into the equation we derived. Plugging (1,3) into y=2x+1 we get (3)=2(1)+1 -> 3=3. Since this is true our solution is correct. (We already know the slope of our equation is correct).

Lines: Point Slope Form Exercises

1) Make use of the slope intercept form of a line to derive equations of the following lines. a) line has slope of -3, and goes through point (-4,2); b) line has slope of 2, and goes through point (2,-4).

2) The line with a slope of m, that passes through the point (x1,y1) is y-y1=m(x-x1). This is the point slope form of a line. Make use of the slope intercept form of a line to derive this equation.

Lines: Point Point Form

We can make use of our ability to determine the slope of a line when we know two of its points, and our knowledge of the point slope form of a line to determine the equation of a line that passes through the two points (x_1,y_1) and (x_2,y_2). (Remember: Always try to solve a problem first, before you look at its solution).

Example Exercise) Determine an equation of the line that contains the two points (-1,-7) and (-8,12).

Solution: First we determine the slope of this line. As explained previously given two points (x_1,y_1) and (x_2,y_2), its slope, m is

$$m = \frac{y_2-y_1}{x_2-x_1}$$

Therefore the slope of the line that passes through the two points P1(-1,-7) and P2(-8,12) is

$$m = \frac{12-(-7)}{-8-(-1)} = \frac{12+7}{-8+1} = \frac{19}{-7}$$

Next we make use of the point slope form of a line to determine the equation of this line. To do this we can make use of either of the points we know of, either (-1,-7) or (-8,12). Lets make use of the point (-8,12).

into the point slope form of a line

$y-y_1=m(x-x_1)$ we substitute the slope m= - 19/7 and the point (-8,12) this gives us

$y-12= (-19/7)\{x-(-8)\}$ or

$y-12= (-19/7)\{x+8\}$ <- This is an equation of the line that passes through the points (-1,-7) and (-8,12).

Lines: Point Point Form Exercises

1) Make use of the point slope form of a line to determine the equation of the line that goes through the points (1,2) (3,-4).

2) a) Calculate the slope, then make use of the point slope form of a line and the point (-1,2) to determine the equation of the line that goes through the points (-1,2) and (5,6). b) Calculate the slope, then make use of the point slope form of a line and the point (5,6) to determine the equation of the line that goes through the points (-1,2) and (5,6). c) Your answers in parts 'a' and 'b' should be the same, are they?

3) Prove: One way to express the line (equation) that passes through the two points (x1,y1) and (x2,y2) is

$$y-y1=\left(\frac{(y2-y1)}{(x2-x1)}\right)(x-x1)$$ <- Point Point form of a line

Lines: Intercept Intercept Form

Looking at a line in intercept intercept form we can by determine by inspection what the x and y intercepts of the line are. x/a + y/b= 1 is a line expressed in intercept intercept form. The x intercept of this line is a and the y intercept form of this line is b, i.e. this line intercepts the x axis at the point (a,0) and this line intercepts the y axis at the point (0,b).

Lines: Intercept Intercept Form Exercises

1) a) Substitute the point (-1,0) into the x/(-1)+y/7=1, now substitute the point (0,7) into this equation. Did doing these substitutions give you an intuitive feel why the intercept intercept line equation 'works as advertised'?

2) I) What are the x and y intercepts of the following lines? II) Graph the following lines. a) x/(-1)+y/2=1; b) x/5+y/1=1; c) x/7+y/(-8)7=1

3) Given the line x/a+y/b=1, prove that its x intercept is 'a' and its y intercept is 'b'.

Lines: Standard Form and General Form

There are two more line forms we need to learn about, they are the standard form of a line and the general form of a line. ax+by=c is a line expressed in standard form. 3x-2y=7 and x+8y=-10 are examples of lines expressed in standard form. ax+by+c=0 is a line expressed in general form. 3x-2y-7=0 and x+8y+10=0 are examples of lines expressed in general form.

Lines: Standard Form and General Form Exercises

1) Name the 7 named line equation forms we discussed and give examples (using numbered coefficients) of each of these forms. Also state in writing what each of these line forms are 'good for'.

Example: Slope Form: $y=3x$; Make use of the slope form of a line if you want a line equation with a particular slope (that passes through the origin).

2) Use algebraic manipulation to put the line $x/8+y/(-1)$ into a) standard form; b) general form.

4) Determine which of the following points are on the line $-2x+10y=14$ a) (1,7)? b) (-2,1)?

5) If the x coordinate of a point on the line $3x-4y=5$ is 10, what is the y coordinate?

6) If the y coordinate of a point on the line $7x+10y=7$ is -7, what is the x coordinate?

7) a) Determine the slope of the line $ax+by=c$; b) Determine one point on the line $ax+by=c$ such that neither coordinate of this point is 0; c) Determine the x and y intercepts of line $ax+by=c$.

8) In the last problem you determined the slope of the line $ax+by=c$. You could have done this problem in either of the following two ways. You could have determined two points of this line, and made use of those two points to determine the slope of this line. Or you could have changed the form of this equation into the slope intercept form and then made use of this equation to determine the slope of this line. a) Make use of the method you didn't use in the previous problem to determine the slope of this line. b) Compare your answer you got in part 'a' with the answer you got in the previous problem. Did you get the same answer both times?

<u>Line Equations Types</u>

Familiarize your self with the following line equation types.

$y=mx$ . . . . (slope) - slope is m, passes through origin
$y=mx+b$ . . . (slope intercept) - slope is m, crosses y axis at b
$y-y1=m(x-x1)$ (point slope) slope is m, passes through point (x1,y1)

$\frac{x}{a}+\frac{y}{b}=1$ . . (intercept intercept)
intersects x axis at a, intersects y axis at b

$$y-y1=\frac{y2-y1}{x2-x1}(x-x1)$$ (two point) - passes through points (x1,y1) and (x2,y2)

ax+by=c . . . (standard form)
ax+by+c=0 . . (general form)

Line Equation Manipulation

Note: The next several exercises include problems asking students to convert one form of line equation into another. It is possible to do this using two different methods. One way is to algebraically manipulate the equation you have into the equation you want. We refer to this method as equation manipulation. The other way is to make use of the equation you are given to determine two points of the line and if needed the slope of the line. Then make use of these things to determine line equation that you desire.

13.6) Slope Intercept y=2x-3

a) I) By inspection, give the slope and the y intercept of this line. II) Verify mathematically that your answer is correct, by verifying that the proposed y intercept is indeed a point of this line, then determine a total of two points that are on this line then use these two points to calculate the slope of this line.
b) Graph this line.
c) I) Find the point of this line that has an x coordinate of 5. Make use of this point to convert the equation into point slope form. II) Check your answer by converting the point slope form of the line, back to the slope intercept form using equation manipulation.
d) I) Convert this equation into intercept intercept form. II) Convert the equation back into its original form using the method you did not make use of in part I of this problem.
e) Convert this equation into standard form.
f) Convert this equation into general form.

7) Point Slope y-8=2(x-3)

a) By inspection give the slope of this line and a point (the most obvious point) that this line passes through. Verify mathematically that these answers are correct.
b) Graph this line.
c) Convert this equation into slope intercept form using equation manipulation.
d) I) Convert this equation into intercept intercept form using equation manipulation. II) Make use of the given equation to determine the x and y intercepts. Make use of these intercepts to determine the intercept intercept form of this equation.
f) Convert this equation into standard form.
g) Convert this equation into general form.

8) Intercept Intercept $\frac{x}{-3} + \frac{y}{7} = 1$

a) By inspection give the x and y intercepts of this line? Verify mathematically that your answers are correct.
b) Graph this line.
c) Convert this equation into slope intercept form using equation manipulation.
d) I) Convert this equation into point slope form using equation manipulation. II) Convert this equation into point slope form, make use of the the two points, the x intercept and the y intercept to do this.
f) Convert this equation into standard form.
g) Convert this equation into general form.

9) Two Point $y-y1=\frac{y2-y1}{x2-x1}(x-x1)$

a) I) If a line goes through the points(1,3)(2,-5), what is the slope of the line? II) What is one obvious point this line goes through? III) Make use of your answers in parts I and II to determine a point slope form of this line.
b) I) What is another obvious point that this line goes through? II) Make use of the slope of this line, along with the point you you mentioned in part I to determine a point slope form of this line. Verify mathematically that your answer in part I agrees with your answer in part II.
c) Graph this line.

In the following problems, d,e,f,g, the words 'this equation' refers to the two point equation above highlighted in yellow. If you find these problems difficult to do, substitute numbers into the parameters (x1,y1,x2,y2) of 'this equation' and then do the problem using this equation. Once you have successfully done this, do these problems d,e,f and g using the two point equation above that is highlighted in yellow. Do the problems below, d,e,f,g using equation manipulation.

d) Convert this equation into slope intercept form.
e) Convert this equation into (intercept intercept) form using a method of your choosing.
f) Convert this equation into standard form.
g) Convert this equation to general form.

10*) General Ax+By+C=0

a) Find the slope of this line; b) Find the slope of this line again using a different method. c) Find the x and y intercepts of this line. d) Find the x and y intercepts of this line again using a different method; Convert the following line equations into general form. e) y=mx; f) y=mx+b;

g) $y-y1=m(x-x1)$; h) $\frac{x}{x1}+\frac{y}{y1}=1$; i) $y-y1=\frac{y2-y1}{x2-x1}(x-x1)$;

j) ax+by=c;

Convert the line Ax+By+C=0 into the following forms
k) slope intercept; l) point slope; m) two point; m) intercept intercept (do two ways); o) standard.

Hint: If you ever have trouble doing a problem because there are variables instead of numbers as parameters, do the problem first by substituting numbers in for the variables, then using this exercise as a guide, do the original problem.

11) Do the following for each of the following lines { a through u } Without looking, give the name of a form of the line equation that most closely corresponds to the information given. (i.e. point slope; intercept intercept; etc). Then find the equation of the line in this most appropriate form.

a) A line crosses the y axis at y=-2 and has a slope of –3;
b) A line passes through the point (-3,-1) and has a slope of –1;
c) A line crosses the x axis at 1, and the y axis at 2;
d) A line passes through points (1,1) and (3,4);
e) A line crosses the y axis at 2 and passes through the point (7,-1);
f) A line crosses the x axis at –1 and passes through the point (6,11);
g) A line crosses the x axis at 5 and has a slope of –3;
h) A line crosses the y axis at –2 and crosses the x axis at 4;
i) A line passes through the point (-2,-5), crosses the x axis at 3;
j) A line has a slope of 2 and crosses the y axis at y=3;
k) A line passes through points, (-1,4) and (3,-8);
l) A line has a slope of 5 and passes through the point (1,2);
m) A line has a slope of 7, and crosses the x axis at –2;
n) A line passes through the point (3,2) and crosses the y axis at –5;
o) A line crosses the x axis at k and has a slope of m;
p) A line crosses the x axis at a, passes through the point(x1,y1);
q) A line crosses the y axis at Yo, passes through the point (x1,y1);
r) A line passes through the points (x1,y1) and (x2,y2);
s) A line crosses the x axis at r, and crosses the y axis at s;
t) A line passes through the point (x1,y1) and has a slope of m;
u) A line has a slope of m and crosses the y axis at k;

Miscellaneous Line Problems

1) a) 2x+3y=12; Determine 2 points that are on this line that are not on either axis. Using this information derive the equation of the line that passes through these two points. Make use of equation manipulation to change this equation to standard form. If the last equation you derived is the same as the given equation, you have done this problem correctly. b) Do this problem again for the line ax+by=c.

2) If one starts at the point (-1,3) and travels from there in a direction of a slope of 7 for some distance, one will arrive at some other point. a) Give an example of such a point; b) Give an other example of another such point, such that (-1,3) is between these two points.

3) Assuming a line has a slope (i.e. the line is not vertical), you can calculate its slope by making use of any two of its points. Make use of the line equation Ax+By+C=0 to prove that it doesn't matter which two points of a line are chosen to calculate its slope, i.e. the calculated slope will be the same regardless of which two of its points are made use of to do the calculation.

4) This is a Highly Recommended Problem
a) A line goes through the points (x1,y1)(x2,y2). Determine the slope of this line and then make use of the slope and of one of these points to determine a point slope form of this line. Next make use of the slope of this line and the other (given) point of this line (the one you didn't make use of previously) to determine a point slope form of this line. You now have two different equations each being a point slope form of this line. Using algebraic manipulation, prove these two equations are the same, i.e. prove these equations are different forms of one another.

Determining Where Two Lines Intersect

If two lines are not parallel, they intersect, (assuming they are coplanar, which is a standard assumption in this book). There are two general methods of determining where two lines intersect. One is to graph both lines and from there you can see the point of intersection. The other way general is to mathematically solve for the point of intersection.

We will now demonstrate two methods to mathematically determine the point of intersection of two lines. The first method makes use of substitution. This method is commonly referred to as the substitution method. The second method makes use of adding two equations together to eliminate one of the variables, (either x or y). The other variable is then solved for by substituting the numerical value of the now known variable into one of the original equations and then solving for the remaining variable. This method is commonly referred to as the elimination method, which doesn't make much sense because both methods eliminate variables.

Substitution Method

Where do the following lines intersect?

line 1: 3x-2y=8
line 2: 7x+3y=5

Solution:

Line 1 ->
3x-2y=8 ->
x=(8+2y)/3

substituting this value of x into the other line, line 2 we get

7[(8+2y)/3] + 3y = 5 -> multiplying both sides by 3 to eliminate fraction(s)
7(8+2y) + 9y = 15 ->
56 + 14y + 9y = 15 ->
23y + 56 = 15 ->
23y = 15 - 56 = -41 ->
y = - 41/23

We can now substitute this value of y into either line 1 or line 2 to solve for x. We choose to make use of line 1.

3x - 2[-41/23] = 8 -> multiplying both sides by 23 to eliminate fraction(s)
69x - 2[-41] = 184 ->
69x + 82 = 184 ->
69x = 184 - 82 ->
69x = 102 ->
x = 102/69 ->
x = 34/23

Therefore the point of intersection of line 1 and line 2 is

```
(34  -41)
(--, ---)
(23   23)
```

It is a good idea to use a calculator to determine if answers to rather complex problems such as this are correct. Having done this, .. we can say this solution is correct.

Elimination Method

Where do the following to lines intersect?

line 1: 3x-2y=8
line 2: 7x+3y=5

Solution

```
3x-2y=8  ->  3*(3x-2y=8)  ->   9x-6y=24
7x+3y=5      2*(7x+3y=5)  ->  14x+6y=10
```

adding these last two equations together we get

23x = 34 ->
x = 34/23

At this point we can make use of the technique we used to solve for x, to solve for y. Instead we will substitute our calculated value of x into either line 1 or line 2 and then solve for y. We choose to make use of line 1.

```
3(34/23)-2y=8 ->  multiplying both sides of this equation by 23
                  to eliminate fraction(s) we get.
3(34)-46y=184 ->
102-46y=184 ->
-46y=184-102 ->
-46y=82 ->
y=82/(-46) ->
y= - 41/23
```

Therefore the point of intersection of line 1 and line 2 is

```
(34 -41)
(--,---)
(23  23)
```

14.4) Lines Intersection Problems
Find the point of intersection of each of the following line pairs using the following methods. I) graphing; II) substitution; III) elimination. a) 4x-3y=11; b) 10x+5y=6

General Points and the Shifting Theorem

The subject of general points and the proof of the shifting theorem, ought be taught in the advanced portion of this book. We are making an exception by putting it here because general points are needed to prove the shifting theorem, and the shifting theorem will be used from here on, in the 'non advanced' portion of this book. Since general points and the proof of the shifting theorem are subjects which ought to be taught in the advanced section of this book, it may be necessary for students to read parts of this subsection two or more times before they fully grasp the material.

A general point is analogous to a variable. A number is one value, but a variable represents a set of values or numbers. Likewise a general point represents a set of points. (a,b) is an example of a general point. (a,b) can represent any point, or every point simultaneously on the Cartesian coordinate system. (a,a^2) is a general point where the y coordinate of the point is equal to the x coordinate squared. Therefore the general point (x,x^2) and the function y=x^2 represent the same set of points as the function.

The way to determine if a point {(2,4) for example} is a member of the set of points represented by an equation {y=x^2 for example} is to substitute the point into the equation. If the resulting equation is true, then the point is a member of the set of points represented by the equation. For example, we substitute the point (2,4) into the equation y=x^2, i.e. we substitute 2 into x and 4 into y, this gives us 4=4 which is true. Therefore the point (2,4) is one of points represented by the equation y=x^2. Substituting (2,5) into y=x^2 gives us 5=4. This is not true, therefore (5,4) is not one of points represented by the equation y=x^2. Likewise if a general point substituted into an equation is true, then the general point is one of the points of the equation. This means that every point represented by general point is also a point of the equation.

For example, {a,f(a)} substituted into y=f(x) is the equation f(a)=f(a). This is a true equation, therefore this general point is a point of the function y=f(x), meaning that every point represented by this general point is a point of this function.

We can say more than this, the point {x,f(x)} is all the points such that the y coordinates of the points equals f(the x coordinates of the points). The function y=f(x) is all the points such that the y coordinates of the points equals f(the x coordinates of the points). Therefore {a,f(a)} and y=f(x) represent the same set of points.

{x+a,f(a)} is the set of points such that f(x coordinate of point -a) = y coordinate of point. y=f(x-a) is the set of points such that f(x coordinate of point-a)= y coordinate of point. Therefore {x+a,f(a)} and y=f(x-a) represent the same set of points.

Q: a) If the function f(x) is shifted to the right by 2, in what direction and by how much is it shifted? b) If the function f(x) is shifted to the right by -2, in what direction and by how much is it actually shifted? c) If the function f(x) is shifted upward by an amount of 2, in what direction and by how much is it shifted? d) If the function f(x) is shifted upward by an amount of -2, in what direction and by how much is it actually shifted?

Notice that the point (2+1,5) is 1 to the right of the point (2,5). Also (2,5+1) is 1 above the point (2,5). Likewise the general point (a+m,b) is m to the right of the point (a,b), the general point (a,b+n) is n above the general point (a,b), and the general point (a+m,b+n) is m to the right and n above the general point (a,b).

We should now have enough familiarity with shifting, functions and general points to responsibly present the proof of the shifting theorem.

Shifting Theorem

If any function f(x) is shifted to the right a distance of u and upward a distance of v, it becomes the function y-v=f(x-u) or y=f(x-u)+v.

Proof

The function y=f(x) and the general point {a,f(a)} each represent (are) the set of points such the y coordinate of the point = f(the x coordinate of the point). Therefore the function y=f(x) and the general point {a,f(a)} represent (are) the same set of points. The function y-v=f(x-u) and general point {a+u,f(a)+v} represents the set of points such that the y coordinate of the point - v = f(the x coordinate of the point - u). Therefore {a+u,f(a)+v} and y-v=f(x-u) are the same set of points. If the set of points {a,f(a)} is shifted to the right a distance of v and upward a distance of u, it becomes the set of points {a+u,f(a)+v). Therefore if the set of points y=f(x) is shifted right a distance of u and upward a distance of v, it becomes the set of points y-v=f(x-u). Therefore any function y=f(x) shifted to the right a distance of u and upward a distance of v, becomes the function y-v=f(x-u) or y=f(x-u)+v.

Proof Complete

Shifting Theorem Problem Set

1) Without looking write the shifting theorem.

2) A set of points is represented by the general point {a,f(a)}. What is the general point representing this set of points if this set of points is shifted a distance of .. a) d to the right? b) d to the left? c) d upward. d) d downward.

3) What is the equation of the function y=f(x) after it has been shifted a distance of a) d to the right? b) d to the left? c) d upward? d) d downward? e) d downward and l to the left?

4) Make use of the slope form of a line and the shifting theorem to derive the equation for the point slope form of a line, i.e. derive the equation of the line that has a slope of m and contains the point (x1,y1).

5) Take the time to understand the proof of the shifting theorem, then prove it for yourself.

Parabolas and Shifting Theorem

Note: If an interesting part of a function is far away from the origin, it might not be desirable for the graph of the function to include the origin. All graphs of functions do not have to include the origin.

1) a) Graph the following three functions on the same graph from x=–4 to 4, where x=-4,-3,-2,...4. y=x^2, y=x^2+1, y=2x^2. b) Which, if any of these functions would you suppose are congruent to each other? Why?

Definition: Parabola
A parabola is any graph that is congruent to the graph y=c*x^2, (c!=0) (!= means not equal). y=x^2 is a parabola. y=ax^2+b (a is not equal to 0) is a parabola.

2) a) Is it apparent that y=x^2 is congruent to y=x^2+b? why? or why not? b) Is it apparent that y=x^2 is congruent to y=x^2+bx? why? or why not?

If student doesn’t understand problem 2, student should graph y=x^2 and y=1 on the same piece of graph paper. Then by looking at these graphs, add them together where x=-4,-3,-2...4. Then connect the points of the function y=x^2+1, with a smooth curved line. Does it seem intuitive (the graph of), y=x^2+1 is congruent (to the graph of) y=x^2? It should because all you have done is moved y=x^2 to a new location, i.e. up by 1. Now imagine graphing y=x^2 and x, and imagine adding these two graphs together in a similar manner. Does it seem as if this graph is congruent to y=x^2? Probably not, because it doesn't seem as if adding the graphs of y=x^2 and y=x together is the same as moving y=x^2 to another location as was the case with adding y=x^2 and y=1. However y=x^2+x and y=x^2 are congruent. y=x^2+mx is congruent to y=x^2. Keep reading and this will be made apparent.

Graph the parabola y=x^2 from -3 to 3. The vertex of this parabola parabola is the point (0,0) and the axis of this parabola is the ray contained in the line line x=0, whose end point is the vertex of the parabola. All parabolas have vertexes and axis, similar to this one.

Definition: Non-Rotated Parabola
A non-rotated parabola, is any parabola where the axis of the parabola is concurrent with, or parallel to the y axis.

3) Make use of the shifting theorem to give the equation of the parabola y=x^2 after it has been shifted 2 to the right and upward by 1.

4) What is the equation of the line y=3x, after it has been shifted 5 units to the right and -4 units upward?

Review Q and A

Q. How is a shifting by a negative amount to be interpreted?
STOP: See if you can answer this question for yourself before reading it below.

A. Any shifting in a certain direction by an amount that is negative, is really a shifting in the opposite direction, by the absolute value of that same amount. For example: A shifting of –2 units to the right is equal to a shifting of 2 units to the left.

5) Determine how the following functions are shifted with respect to each other. a) y=x vs y-1=x+2; b) y=3x vs y+2=3(x+7); c) y=3x vs y=3x+21; d) y=ax vs y=a(x-b); e) y=mx vs y=mx+b; f) y=x^2; vs y-4=(x+1)^2; d) y=-2x^2 vs y-5=-2(x-6)^2; e) y=f(x) vs y+3=f(x-2)

6) Make use of the shifting theorem to find equations of the following functions. a) y-3=3(x-1), shifted 2 units up and 3 units to the right. b) 2y-1=5x+1, shifted 4 units up and 7 units to the left. c) 2y+3=(-3x+4)^2, shifted 2 units left and 6 units down; d) y=3x shifted 1 unit to the right and 2 units down.

7) The equation y-y0=m(x-x0) is in point slope form. Make use of the shifting theorem to show (prove) this line contains the point (x0,y0).

8) Making use of the shifting theorem, one can see that shifting the line y=2x to the right one unit is the same as shifting this line 2 units downward. a) Show this is true graphically. b) Prove this is so analytically. c) Derive a similar rule for the line y=cx?

9) I) Determine the vertex and the axis of the following parabolas. II) Make use the shifting theorem to help graph the following parabolas. III) Plot points to help verify your answers.
a) y+1=2(x-3)^2; b) 4y=(6x-8)^2+4;

10) By making use of the method of "completing the square", it is possible to prove that the graph of y=x^2 is congruent to the graph of y=x^2+2x. (STOP: Try to do this on your own before looking at the proof below).

Proof that y=x^2+2x is congruent to y=x^2.

```
y=x^2+2x  ..  (completing the square) is the same function as
y+1=x^2+2x+1  ..  is the same function as
(y+1)=(x+1)^2  .. this function is congruent to y=x^2 because this
                  function is the function y=x^2 after it has been
                  shifted 1 to the left and 1 down.
```

Therefore y=x^2+2x is the function y=x^2 after it has been shifted to a new location. Therefore the function y=x^2+2x is congruent to the function y=x^2.

If you weren’t able to do the preceding proof on your own without looking, study this proof then do it now, (without looking).

11) Now as promised, it will be made apparent (proved) that y=x^2+bx is congruent to y=x^2. Try prove this yourself before looking at the proof below.

Proof that y=x^2+bx is congruent to y=x^2

```
y=x^2+bx  ..  is the same function as


  ( b )2        ( b )2
y+(---) = x^2+bx+(---)  ..  is the same function as
  ( 2 )         ( 2 )


    b^2   (     b )2
y + --- = (x + ---)  ..  and this function is congruent to y=x^2
     2    (     2 )      because this function is y=x^2 after it
                         has been shifted b/2 to the left and
                         (b^2)/2 down.
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Therefore $y=x^2+bx$ is the function $y=x^2$ after it has been moved to a different location. Therefore the functions $y=x^2$ and $y=x^2+bx$ are congruent.

If you weren't able to do this proof on your own without looking, do it now, .. without looking.

The standard form of a parabola is $y-y1=m(x-x1)$

12) Any two functions, $y=f(x)$ and $y=g(x)$, added together are $y=f(x)+g(x)$. $y=1$ added to $y=x$ is $y=x+1$. a) Prove that the parabolas $y-2=3(x+1)$ and $y=x^2$ added together are a parabola. b) Prove that any non-rotated parabola, added to any other non rotated parabola, is either a non-rotated parabola or a line. c) Under what conditions is the sum of two non-rotated parabolas a parabola? A line? c) What parabola must be added to $y=x^2$ in order that the sum of these parabolas will equal $y=-(x-2)^2+1$?

Note: Because it is relatively easy to show analytically that any two non-rotated parabolas added together are either a line or a parabola, the significance of this may be easy to overlook. Geometrically this is a rather amazing fact. Take the time and effort to appreciate this from a geometrical point of view.

13) a) Determine the function that has to be added to $y=x^2$ to shift it 2 units to the right. b) Using a scale of 2 horizontal squares equals one unit, and one vertical square equals one unit, graph $y=x^2$ and the function calculated in a) on the same piece of graph paper. c) Graphically add these two functions together then judge if this resultant function is $y=x^2$ shifted 2 units to the right. Do this from $x=-4$ to $x=6$, where $x=-4,-3,-2...6$.

17.13) a) Shifting $y-5=-2(x+3)^2$, 1 to the right and up 2 produces the same result as adding what function to this parabola?

15) a) Prove that adding a non-vertical line to any non-rotated parabola results in a parabola, that is congruent to the original parabola but shifted to a new location. b) Prove that any non-rotated parabola may be shifted to any desired position by adding to it, an appropriate line.

Note: The following two facts (were proved in the last problem)

I) A non vertical line added to a non-rotated parabola results in an other parabola that is congruent to the original parabola, but shifted to a different location.

II) Any non-rotated parabola can be shifted to any desired location by adding to it an appropriate line.

These are quite amazing geometrical theorems, take the time to appreciate these facts from a geometrical point of view.

15) Suppose a the vertex of a parabola congruent to $y=x^2$ is moved from (5,4) to (-3,-1). What line do we need to add to the parabola in order to accomplish this?

16) a) Prove that any non-rotated parabola in standard form, i.e. $y-y1=d(x-x1)^2$, can be written as $y=ax^2+bx+c$. b) Prove the function $y=ax^2+bx+c$ can be written as a parabola in standard form. We have now proven that any parabola can be represented as $y=ax^2+bx+c$.

17) **The general form of a parabola** is $y=ax^2+bx+c$. Convert the parabola $ay+k=(mx+q)^2$ to a) standard form. b) general form.

18) I) Determine the vertex and the axis of the following parabolas. II) Make use of your answers in part I to help graph these parabolas. a) $y=x^2+8x-9$; b) $y=2x^2+18x+28$; c) $y=-x^2+18x-32$

19) Leave the equations of the shifted parabolas in the same form as the original parabolas. a) What would be the equation of the parabola $x^2+6x-10$ if it were moved 3 units left and 1 unit upward? b) What would be the equation of the parabola $y=3x^2+12x-30$ if it were shifted 2 units downward and 1 unit to the right? c) What would be the equation of the parabola $3y=(2x+1)^2+6$ if it were moved 1 unit up and 2 units left?

20) Find where the following function pairs intersect. a) $y=3x+7$, $y=x^2-5$; b) $y=x^2+x-1$, $y=-3x^2+12$;

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3 Parallel Lines / Perpendicular Lines

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In this section we study slope of parallel and perpendicular lines. We prove theorems that allow us to determine if lines are parallel or not depending upon their slopes. We also develop theorems which allow us to determine if lines are perpendicular or not depending on their slopes.

Parallel Lines

Parallel Lines Definition:
Lines are parallel if they are coplanar and do not intersect.

Keep in mind that vertical lines have undefined slope, and lines with undefined slope are vertical. This stems from the fact that if a line is vertical, change of x is 0 and
slope = (change y)/(change x).

Parallel Line Theorems

a) Non vertical, parallel lines .. have the same slope.
b) Lines with the same slopes .. are parallel.
c) Non vertical, non parallel lines .. have different slopes.
d) Lines with different slopes .. are not parallel.
e) If lines are parallel .. and one of them is vertical, the other line is also vertical.
f) Vertical lines .. are parallel.

Assuming one is willing to accept the following terminology, the parallel line theorems can be simplified.

I) Two lines, each with undefined slope have the same slope.
II) If one line has a defined slope and another line has an undefined slope then these lines have different slopes.

Parallel Line Theorems (Simplified) <- A Recommended Version to keep in mind.

a) Parallel lines have the same slope.
b) Lines with the same slope are parallel.
c) Non parallel lines have different slopes.
d) Lines with different slopes are not parallel.

Parallel Line Theorems (Extremely Simplified) <- Most Recommended

a) Lines are Parallel <-> Lines have the same slope
b) Lines are not Parallel <-> Lines have different slopes

It is left to the student to prove the Parallel Line theorems (simplified) by making use of the Parallel Line theorems (non simplified) if they wish. Proofs of the Parallel Line theorems (non simplified) follow.

Theorem d): Lines with different slopes are not parallel.

Proof:

Line 1: $y-y_1= m_1(x-x_1)$ and line 2: $y-y_2= m_2(x-x_2)$ represent every possible set of lines with different slopes. Therefore if it can be proven (where $m_1 \neq m_2$) that these two general lines are not parallel, (i.e. they do intersect), the theorem is proven.

The definitions of line 1 and line 2 imply the following set of equations.

$$y=y_1+m_1(x-x_1) : y=y_2+m_2(x-x_2) \rightarrow$$

$$y_1+m_1(x-x_1) = y_2+m_2(x-x_2) \rightarrow$$

$$y_1 + m_1 x - m_1 x_1 = y_2 + m_2 x - m_2 x_2 \rightarrow$$

$$m_1 x - m_2 x = y_2 - y_1 + m_1 x_1 - m_2 x_2 \rightarrow$$

$$x(m_1-m_2) = y_2 - y_1 + m_1 x_1 - m_2 x_2$$

$$x = \frac{y_2 - y_1 + m_1 x_1 - m_2 x_2}{m_1-m_2}$$

Therefore these lines intersect where x is the value shown above, and $y=y_1+m_1(x-x_1)$. Therefore these lines do intersect. Therefore these lines are not parallel.

Proof Complete

Parallel line theorems a,b and c can also be proven by making use of a slight variation of the preceding proof. In the problem section a problem will be given asking students to prove parts a,b,c and d this way. A solution to these problems is provided in solutions section. In the next few pages we prove parts a and c using a different method. This method is more difficult and less intuitive, but has the advantage of giving students the opportunity to see and practice indirect proofs. As always, try to do the following proof yourself before looking at it.

Theorem b): Lines with the same slopes .. are parallel.

Proof:

Line 1: $y-y_1= m_1(x-x_1)$ and line 2: $y-y_2= m_2(x-x_2)$ represent every possible set of lines with (potentially) different slopes. (If $m_1 = m_2$, these lines have the same slope). Therefore if it can be proven that $m_1 = m_2$, these two general lines are parallel, (i.e. they do not intersect), the theorem is proven.

The definitions of line 1 and line 2 imply the following set of equations.

$$y=y_1+m_1(x-x_1) : y=y_2+m_2(x-x_2) \rightarrow$$

$$y_1+m_1(x-x_1) = y_2+m_2(x-x_2) \rightarrow$$

$$y_1 + m_1*x - m_1*x_1 = y_2 + m_2*x - m_2*x_2 \rightarrow$$

$$m_1*x - m_2*x = y_2 - y_1 + m_1*x_1 - m_2*x_2 \rightarrow$$

$$x(m_1-m_2) = y_2 - y_1 + m_1*x_1 - m_2*x_2$$

$$x = \frac{y_2 - y_1 + m_1*x_1 - m_2*x_2}{m_1-m_2}$$

Therefore these lines intersect where x is the value shown above, If $m_1 = m_2$, the denominator is 0, implying that if $m_1 = m_2$, there is no x (real numbered) value for which these lines intersect. Given that $y=y_1+m_1(x-x_1)$, there is no y (real numbered) value for which these lines would intersect either. Therefore if $m_1 = m_2$, these lines do not intersect (in the Cartesian coordinate system) and these lines are parallel.

Proof Complete

The following three proofs are indirect proofs. When doing an indirect proof, you make an assumption, if this assumption can lead to a contradiction, the assumption is then proven to be false. This kind of proof works because true assumptions do not lead to contractions. [The previous sentence is a postulate]. Take a few moments to see for yourself, that true assumptions do not lead to contradictions.

Theorem: c) Non vertical, non parallel lines .. have different slopes.

Proof: I) Assume that non vertical, non parallel lines .. can have the same slope. II) A and B are non vertical non parallel lines, statement I therefore implies they can have the same slope, lets assume they do. III) Theorem b states, [Lines with the same slopes .. are parallel]. This implies that lines A and B are parallel.

According to statement II lines A and B are not parallel. Statement III says lines A and B are parallel. This is a contradiction, therefore the initial assumption, i.e. statement I leads to a contradiction and is therefore not true. Therefore 'Non Vertical, non parallel lines .. have different slopes.

Proof Complete

Theorem: a) Non vertical, parallel lines .. have the same slope.

Proof: I) Assume that Non vertical, parallel lines .. can have different slopes. II) A and B are non vertical, parallel lines, therefore statement I implies they can have different slopes, lets assume they do. III) By theorem 'd', [Lines with different slopes .. are not parallel], lines A and B are not parallel.

According to statement II, lines A and B are parallel. According to statement III, lines A and B are non parallel. This is a contradiction. Therefore the initial assumption I is not true. Therefore non vertical, parallel lines .. have the same slope.

Proof Complete

Theorem e): If lines are parallel .. and one of them is vertical .. the other line is also vertical.

I) Lines A and B ARE parallel and A IS vertical. Since A is vertical, A can be represented by the equation x=a. II) Lets assume B is not vertical. This implies B can be represented by the equation y=mx+b. Therefore we have the following two equations, one representing any vertical line and the other representing any non vertical line.

```
x=a : y=mx+b  ->
x=a : x=(y-b)/m  ->
a = (y-b)/m  ->
a*m = y-b  ->
y=a*m+b
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Therefore lines A and B intersect where y=a*m-b and where x=a, or at the point (a,a*m-b). Therefore the lines A and B do intersect.

Therefore lines A and B are not parallel. Our assuming statement II has allowed us to 'prove' A and B are parallel. However this contradicts statement I which states that A and B ARE parallel. (Statement I is not an assumption, Statement I IS true). Therefore statement II is not true, therefore B is vertical.

Proof Complete

Theorem f): Vertical lines .. are parallel.

A is a vertical line, therefore A can be represented by the equation x=a. B is a vertical line, therefore B can be represented by the equation x=b. Therefore the following two equations represent any two vertical lines.

x=a : x=b -> .. Solving where these two lines intersect we have a=b

Therefore lines A and B intersect only at a place where a=b. Therefore if a != b, lines A and B do not intersect, implying that A and B are parallel. If a=b, lines A and B are the same line because A and B could then both be represented by the same equation, .. x=a. Given that A and B are distinct lines, (which by unstated assumption they are), they do not intersect and therefore are parallel. Therefore vertical lines are parallel.

Proof Complete

Perpendicular Lines

Perpendicular Line Definition:
Two lines are perpendicular if they contain a pair of rays that form a 90 degree angle.

Perpendicular Lines Theorems

a) If lines 1 and 2 are perpendicular, and line 1 has slope m1, and line 2 has slope m2, then m1*m2= -1;

b) If line 1 and line 2 have slopes m1 and m2 respectively, and if m1*m2=-1, then line 1 and line 2 are perpendicular.

c) If a line has a slope of 0 and another line is perpendicular to it, then this other line has an undefined slope, (is vertical).

d) If a line has an undefined slope (is vertical) and another line is perpendicular to it, then this other line has a slope of 0.

e) If a line has a slope of 0, and another line has an undefined slope (is vertical), then these two lines are perpendicular.

Theorem 'a': If the lines L1 and L2 are perpendicular, and L1 has slope m1 and L2 has slope m2, then m1*m2=-1

The Picture

(It is important to remember that this picture represents a general situation, and is not meant to represent a picture only of: L1 with slope 1/2 and L2 with slope of -2).

First draw a Cartesian coordinate system. Draw a line (L1) so that it has a slope 1/2. Draw a line (L2) so that it has a slope of about –2 and intersects L1 at a point P. Mark a point A on L1 above and to the right of P. Mark a point B on L2 above and to the left of P such that PA=PB. Let L be the horizontal line through P. From A drop a perpendicular to L and call the point where this perpendicular meets L, A'. From B drop a perpendicular to L and call the point where this perpendicular meets L, B'. [The symbol < means angle].

Proof

a) L1 and L2 are perpendicular - given
b) PA=PB - given
c) BB' and AA' are both perpendicular to L - given
d) <B'PB = <A - both angles are complimentary to angle APA'
e) <BB'P = <AA'P = 90 degrees - def of perpendicular (see step c)
f) Triangles BB'P and AA'P are congruent - AAS (see steps b,d,e)
g) PA' = BB' = I - CPCTC - (see steps f and d)
h) PB' = AA' = II - CPCTC -

i) $$\text{slope L1} = \frac{\text{rise of L1}}{\text{run of L1}} = \frac{AA'}{PA'} = \frac{II}{I}$$

j) $$\text{slope L2} = \frac{\text{rise of L2}}{\text{run of L2}} = \frac{BB'}{-PB'} = \frac{I}{-II}$$

k) $$\text{slope L1} * \text{slope L2} = \frac{II}{I} * \frac{I}{-II} = -1$$

Proof Complete

Theorem 'b': If lines L1 and L2 have slopes m1 and m2, where m1*m2=-1, then L1 and L2 are perpendicular

The Picture

(It is important to remember that this picture represents a general situation, and is not meant to represent a picture only of: L1 with slope 1/2 and L2 with slope of -2).

First draw a Cartesian coordinate system. Draw a line (L1) so that it has a slope 1/2. Draw a line (L2) so that it has a slope –2 and intersects L1 at a point P. Mark a point B on L2 above and to the left of P. L is the horizontal line through P. From B drop a perpendicular to L. B' is the point where this perpendicular and L meet. Mark a point A on L1, above and to the right of P such that the length of the perpendicular from A to L equals B'P. A' is the point where this perpendicular and L meet.

Proof

a) slope of L1 * slope of L2 = -1 .. given
b) AA' and BB' are both perpendicular to L .. given
c) AA' = B'P = I .. given
d) <AA'P = <BB'P = 90 degrees - property of perpendicular - (see b)

$$\text{e) slope of L1} = \frac{\text{rise of L1}}{\text{run of L1}} = \frac{AA'}{PA'} = \frac{I}{PA'}$$

$$\text{f) slope of L2} = \frac{\text{rise of L2}}{\text{run of L2}} = \frac{BB'}{-PB'} = \frac{BB'}{-I}$$

g) since slope of L1 * slope of L2 = -1 it follows that

$$\frac{I}{PA'} * \frac{BB'}{-I} = -1 \quad -> \quad \frac{BB'}{-PA'} = -1 \quad -> \quad BB' = PA' = II$$

h) Triangles PAA' and PAA' are congruent .. SAS (see steps c,d,g)

i) <B = <APA' = I' -CPCTC- (both angles opposite sides of length I)
j) <A = <BPB'= II' -CPCTC- (both angles opposite sides of length II)

k) <B + <BPB' = 90 degrees - these angles are two of the angles of the triangle PBB'. The other angle of this triangle is <BB'P which is 90 degrees. (see step d)

l) <I' + <II' = 90 degrees .. angles I' and II' are angles B and BPB' respectively, and <B+<BPB'=90 degrees. (see steps i,j,k)

m) <APA'+<BPB'=90° .. <APA'=I' and <BPB'=II and I'+II'=90°

m) <BPA, (an angle formed by L1 crossing L2) is a 90 degree angle.
.. <BPA + <B'PB + A'PA = 180° -> <BPA + I' + II' = 180° ->
<BPA+90° = 180° -> <BPA = 90°.

Therefore L1 and L2 are perpendicular

Proof Complete

The proofs of the perpendicular line theorems, parts c,d,e are not provided at this time.

<u>Parallel Lines & Perpendicular Lines Problem Set</u>

1.1) Given the points E(-4,0), G(3,5), I(15,3) and K(8,-2). a) Show that lines GE and KI are parallel by showing that their slopes are equal. b) Show that lines EG and GK are perpendicular by showing the product of their slopes is –1. c) Check your answers graphically, by graphing these lines.

2) L1, L2, L3, L4, L5 are lines with slopes 2/3, -4, -1 1/2 = -1.5, 1/4, -4 respectively. a) Which of these line pairs are parallel? b) Which of these line pairs are perpendicular? c) Check your answers graphically, by graphing lines with these slopes.

3) Consider the points (-2,-12)(2,1)(4,-11)(14,3). a) Which lines determined by these points are parallel? b) Which lines determined by these points are perpendicular?

1.4) The vertices of a triangle are (16,0), (9,2), and (0,0). a) What are the slopes of its sides? b) What are the slopes of its altitudes? c) What are the equations of the lines containing the altitudes of this triangle?

5) Prove that the quadrilateral with vertices (-2,2), (2,-2), (4,2), and (2,4) is a trapezoid with perpendicular diagonals.

6) a) If the line containing points (-8,m) and (2,1) is parallel to the line containing points (11,-1) and (7,m+1), what is the value of m? b) Do part 'a' again, assuming the lines are perpendicular.

1.7) Do "a" and "b" graphically before doing them analytically. What value(s) of k will make the lines (k,3)(-2,1) and (5,k)(1,0) a) parallel? b) perpendicular?

8) Do this problem graphically first, then analytically. Given the points P(1,2), Q(5,-6), and R(b,b), determine the value(s) of b so that <PQR is a right angle.

9) Find the equation of the line, that is parallel to the line y=3x+2 and contains the point (2,-3).

10.1) Find the equation of the line, that is perpendicular to the line y=3x+2 and contains the point (2,-3).

11.1) a) Two slopes are perpendicular and one of them can be represented by the expression –x+3 and the other one can be represented by the expression 9x-1, where x is some yet unknown value. What are each of these slopes? (If there exists two sets of slopes, find both). b) Do part 'a' again assuming the two slopes are parallel, not perpendicular.

12$) Make use of line equations to prove each of the following.

a) Non vertical, parallel lines .. have the same slope.
b) Lines with the same slopes .. are parallel.
c) Non vertical, non parallel lines .. have different slopes.
d) Lines with different slopes .. are not parallel.

13) Make use of classical geometry to prove: If lines 1 and 2 are perpendicular, and line 1 has slope m1, and line 2 has slope m2, then m1*m2= -1; (m1 and m2 are negative reciprocals of each other)

14) Make use of classical geometry to prove: If line 1 and line 2 have slopes m1 and m2 respectively, and if m1*m2=-1, then line 1 and line 2 are perpendicular.

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4 Distance Formula

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The distance formula is one of the fundamental theorems of coordinate geometry. When given the any two points, the distance formula gives the distance between the two points. Given the points P1(x1,y1) and P2(x2,y2), the distance between these two points is d(P1,P2)=sqrt{(x2-x1)^2+(y2-y1)^2}, this is the two dimensional distance formula.

The 1 dimensional distance formula is required to derive the two dimensional distance formula. We do not prove the 1 dimensional distance formula, it is accepted as a postulate. When given the coordinate(s) of two points, it gives the distance between these points when the line connecting the two points is either vertical (parallel to the y axis) or horizontal (parallel to the x axis). Examples of the one dimensional distance formula are given below.

Where 'a' and 'b' are points (or numbers) on a number line (or the x axis). The distance between 'a' and 'b' is |b-a|, which in words is the absolute value of b-a.

Where A(x1,y1) and B(x2,y1) are two points, the distance between A and B is |x2-x1|. Where A(x1,y1) and B(x1,y2) are two points, the distance between A and B is |y2-y1|.

Take the time to understand for yourself that the 1 dimensional distance formulas are true. This will require you to draw pictures. Be able to convince someone besides yourself that these formulas are true.

The derivation of the 2 dimensional distance formula is given below. As always see if you can do a derivation on your own without looking at the given derivation.

Derivation of 2 Dimensional Distance Formula

The Picture
Let P1(x1,y1) and P2(x2,y2) be two points on the Cartesian coordinate system. Let L be the horizontal line (parallel to the x axis) through P1. Let P be the point on L such that the segment P2-P is perpendicular to L. P then is the point (x2,y1).

a) segment PP2 is perpendicular to segment PP1 .. given
b) <P1-P-P2 is a right angle .. implied by step a.
c) P1-P-P2 is a right triangle .. implied by step b.
d) (PP1)^2 + (PP2)^2 = (P1P2)^2 .. Pythagorean theorem
e) Length of segment PP1 is |x2-x1| .. 1 d distance formula
f) Length of segment PP2 is |y2-y1| .. 1 d distance formula
g) length of P1P2 = sqrt{(x2-x1)^2+(y2-y1)^2} .. substitute |x2-x1| of step 'e' into PP1 of step 'd'. substitute |y2-y1| from step 'f' into PP2 of step 'd'. Then take the square root of both sides of the resulting equation.

Derivation complete

Example Problems
The problems of this section that have solutions in the back of the book are .. 1,4,7,10,11 and 15. These problem may serve as example problems if needed. Otherwise the student may do these problems on their own. The fact that these problems have solutions in the back of the book is made evident by the numbering format used. These problem are numbered as 1.1, 1.4, 1.7, 1.10, 1.11 and 1.15.

Distance Formula Problem Set

2.1) Use the distance formula to find the distance between the following points. a) (0,0) and (3,4); b) (1,2) and (6,14); c) (5,-1) and (-3,5).

2) a) Find the perimeter of a triangle whose vertices are (5,7), (1,10), and (-3,-8); b) Find the area of this triangle.

3) Given points A(-5,4), B(3,5), C(7,-2), D(-1,-3). Prove that quadrilateral ABCD is a rhombus.

2.4) a) Find the distance between the point (1,7) and the line y=3x; b) Find the distance between the point (a,b) and the line y=cx+d.

5) a) Find the distance between the lines a) y=3x and y=3x+7; b) y=mx and y=mx+b; c) y=mx+b and y=mx+c. (Hint: The distance between two lines is length of shortest path between them).

6) Find the value(s) of b such that the triangle whose vertices are (-6,0),(0,6) and (b,-b) is equilateral.

2.7) R is the ray (0,0)(2,6) whose end point is the origin. a) Without making use of the distance formula, propose a point on R that is twice as far from the origin as (2,6), then make use of the distance formula to check your answer. b) Without making use of the distance formula, propose a point on R that is half as far from the origin as (2,6). c) Using the distance formula, determine the distance of (2,6) from the origin, then using the same method used in a and b propose a point on R that is a distance of 1 from the origin. d) Propose a point on R that is a distance 7 from the origin.

8) (Do problem 7 before doing this problem). A ray's endpoint is on the origin, it has a slope of m and lies in the first quadrant. Determine the point on this ray that is a distance of k from the origin.

9) If (3,k) is equidistant from A(2,-3) and B(7,4), what is k?

2.10) The following sets of points are vertices of rhombus given in counter clockwise order, starting with the lower left vertex, then lower right, then upper right, then upper left. Make use of the distance formula to determine all vertices. Do not make use of vectors when doing this problem. In Appendix 2 vectors will be studied, this problem is easier to do when using vectors.
a) (0,0)(1.9225,y2)(3.1261,2.1486)(x4,1.5973);
b) (3,2)(8,2)(11,6)(x,y).

2.11) Derive 3 Dimensional (3D) Distance Formula
Make use of the 2D distance formula to derive the 3D distance formula, i.e. show that the distance between P(x1,y1,z1) and Q(x2,y2,z2) is given by the formula
d = sqrt{(x2-x1)^2+(y2-y1)^2+(z2-z1)^2}.

12) Find the distance from the origin to the point P(a,b,c)?

13) Find the distance between the following sets of points.
a) P(4,-1,-5), Q(7,1,7); b) P(0,4,5), Q(-6,2,3).

14) Prove that the triangle with vertices (2,0,8), (8,-4,6) and(-4,-2,4) is isosceles.

2.15) Prove triangle A(4,-2,3) B(1,0,4) C(7,10,-12) is a right triangle.

16) The figure ABCD has vertices A(3,2,5), B(1,1,1), C(4,0,3), and D(6,1,7). A) Show that its opposite sides are congruent.
b) Is ABCD necessarily a parallelogram?

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5 Midpoint Formula & Proportional Point Formula
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Midpoint Formula

Midpoint Formula
When given the endpoints of a segment, the Midpoint Formula gives the point of the segment that is equidistant from the endpoints of the segment. This point is called the midpoint of the segment.

- 1 Dimensional Midpoint Formula -

Consider the points (or numbers), a and b, on the number line. Guess a formula which will provide the midpoint m the segment ab. The answer follows, try to do this yourself before looking.   STOP

Answer: It seems reasonable to choose the point (number) that is the average of a and b and this is what we do, i.e. midpoint of ab = m = (a+b)/2. Now prove that (a+b)/2 is the midpoint of ab. A proof follows. Try to do this yourself before looking.   STOP

This formula is true if am=mb, meaning if m-a=b-m. Substituting (a+b)/2 in for m we have [(a+b)/2]-a = b-[(a+b)/2]. This equation simplifies to a/2 + b/2 -a = b - a/2 - b/2 which simplifies to (b-a)/2 = (b-a)/2, therefore the proposed formula is true.

If you were not able to derive the 1 dimensional midpoint formula on your own, study this derivation, then do it, without looking.

- 2 Dimensional Midpoint Formula -

Make use of the 1 dimensional midpoint formula and similar triangles to derive a formula for the midpoint of a segment in 2 dimensions. A derivation follows. Try to derive this formula on your own without looking at the derivation below.   STOP

Derivation of 2 Dimensional Midpoint Formula

Consider the segment A(x1,y1) B(x2,y2), where y2>y1 and x1 is not equal to x2. Let P(?,??) be the midpoint of this segment. We introduce the point C(x2,y1) directly below point B and horizontal to A. We introduce the point D directly below the point P and on segment AC. We also introduce a point E, horizontal to P and on segment BC.

We note that triangles ABC and APD are similar. Therefore AP/AB=1/2, implies AD/AC=1/2 and PD/BC=1/2. Therefore D is the midpoint of AC, and E is the midpoint of BC.

Applying the 1 dimensional midpoint formula to AC we have ? = (x1+x2)/2. Applying the 1 dimensional midpoint formula to BC we have ??= (y1+y2)/2. Therefore the coordinates of P, the midpoint of AB are {(x1+x2)/2,(y1+y2)/2}.

If you were unable to do this derivation on your own, without looking, study the derivation, then do it on your own without looking.

- Midpoint Formula Classical Form -

p1=(x1,y1) and p2=(x2,y2), the midpoint m of the segment p1 p2 is

```
    [ x2+x1   y2+y1 ]
m = [ ----- , ----- ]   <-----   midpoint formula classical form
    [   2       2   ]
```

- Midpoint Formula Vector Form -

A=(x1,y1) and B=(x2,y2), A+B=(x1+x2,y1+y2)

where m, A and B are points, the midpoint m of the segment A B is

```
      A+B
m =   ---   <----  midpoint formula vector form
       2
```

Proportional Point Formula (PPF)

PPF (Proportional Point Formula)
When given the endpoints of a segment, a Proportional Point Formula gives the point p of segment that is the fraction c of the way from one of the end points towards the other end point.

If (0<c<1), then p will be on the segment defined by the endpoints.

For example if we wanted to find the point of the segment (-1,2)(4,7) that is 5/9 of the way from (-1,2) towards (4,7) we would use a proportional point formula to find this point.

- 1 Dimensional Proportional Point Formula -

Consider the points (or numbers) a and b on the number line. Guess a formula which will provide the point that is 2/5 (or some other fraction) of the way from point a, towards the point b. The answer follows, try to do this yourself before looking. STOP

If someone starts at point a on the number line and wishes to go ALL THE WAY to the point b, what would they add to a? Answer is (b-a), i.e. a+(b-a) is b. It seems reasonable then, if someone starts at the point a and wants to go ..2/5.. of the distance towards b, they would add (2/5)(b-a) to a. The equation we propose then is
p = a+(2/5)(b-a).

Now prove that p=a+(2/5)(b-a) is indeed 2/5 of the way from point 'a' towards point 'b'. The proof follows. Try to do this proof on your own before looking. STOP

This formula is true if p-a = 2/5(b-a), substituting a+(2/5)(b-a) into p, this becomes [a+2/5(b-a)]-a = 2/5(b-a). This equation simplifies to 2/5(b-a) = 2/5(b-a). Therefore the proposed formula is true.

If you were not able to derive the 1 dimensional proportional point formula on your own, study this derivation, then do it, without looking.

- 2 Dimensional Proportional Point Formula -

Make use of the 1 dimensional proportional point formula and similar triangles to derive a 2 dimensional proportional point formula. A derivation follows. Try to do this on your own, before looking below. STOP

Derivation of 2 Dimensional Proportional Point Formula

Consider the segment A(x1,y1) B(x2,y2), where y2>y1 and x1 is not equal to x2. Let P(?,??) be the point of this segment that is the fraction k, from the point A towards the point B. We introduce the point C(x2,y1) directly below point B and horizontal to A. We introduce the point D directly below the point P and on segment AC. We also introduce a point E, on segment BC and horizontal to point P.

We note that triangles ABC and APD are similar. Therefore AP/AB= k implies AD/AC = k and PD/BC = k. Therefore D is the point on AC that is the fraction k from the point A, towards the point C, and E is the point on BC that is the fraction k from the point the point C towards the point B.

Applying the 1 dimensional proportional point formula to AC we have ? = x1 + k(x2-x1). Applying the 1 dimensional proportional point formula to BC we have ?? = y1 + k(y2-y1). Therefore the coordinates of P which is the point that is the fraction k from (x1,y1) towards the point (x2,y2) are {x1+k(x2-x1),y1=k(y2-y1)}. If you were unable to derive this formula on your own without looking, then study this derivation now, then do it on your own without looking.

- Proportional Point Formula Classical Form -

where p1=(x1,y1) and p2=(x2,y2)

if p is the point on the segment p1 p2 such that $\dfrac{p1\ p}{p1\ p2} = r$

then

p={x1+r(x2-x1),y1+r(y2-y1)}   <---   proportional point formula classical form

- Proportional Point Formula Vector Form -

Definition: Point (or Vector) Addition
P1(x1,y1) + P2(x2,y2) = (x1+x2,y1+y2)

A and B are points

if p is the point on the segment A B such that $\dfrac{A\ p}{A\ B} = r$

then

p=A+r(B-A) <----   proportional point formula vector form

Note: If (0<r<1), r will be between p1 and p2, or between A and B.

Example Problems
The problems of this section that have solutions in the back of the book are .. 1,2,6,7,9,12,13,14,16. These problem may serve as example problems if needed. Otherwise the student may do these problems on their own. The fact that these problems have solutions in the back of the book is made evident by the numbering format used. These problem are numbered as 3.1, 3.2, 3.6, 3.7, 3.9, 3.12, 3.13, 3.14, 3.16. Please read the Preface in the front of the book to learn more about problems which have solutions in the back of the book and how to easily access these answers.

Midpoint Formula & Proportional Point Formula Problem Set

3.1) I) Use the midpoint formula to find the midpoint of the following segments. II) Use the proportional point formula to find the midpoint of each of the following segments.
a) (6,0) (10,2); b) (a,b) (c,d).

3.2) What is the trisection point of the segment (2,-3) (8,9)
a) closest to the point (2,-3)? b) closest to the point (8,9)?

3) Use the proportional point formula to derive the midpoint formula.

4) Given the segment A(x1,y1) B(x2,y2): a) If someone starts at A and then goes the fraction c of the way towards B, what is the coordinate of the point where they end up? b) if someone starts at B and then goes the fraction 1-c of the way towards A, where do they end up?

5) If A(3,15) and C(13,0) are the end points of a segment and B is a point on segment AC: Find the coordinates of B given that the ratio of AB/AC is a) 5/7; b) 2/5; Find the coordinates of B given that the ratio AB/BC is c) 4; d) 2/3; Find the coordinates of B given that the ratio of BC/AB equals e) 3/8; f) 12/13.

3.6) Given G(-5,8), K(2,a), H(b,1). a) Find a and b so that K will be the midpoint of GH; b) Find a and b so K will be 3/7 of the way from H towards G.

3.7) a) A segment has midpoint M(3,-5), and one end point is A(2,-4). What are the coordinates of the other end point?

8) A segment has an end point of (-2,7), the segment trisection point closest to this end point is (1,4). What is the other end point of the segment?

3.9) Prove that two of the medians of the triangle with vertices (-m,0), (m,0), and (0,3m) are perpendicular to each other.

10) a) If one starts at point (x1,y1) and goes 2/7 of the way towards (x2,y2) they will be at (-7,-1). If one starts at (x2,y2) and goes 1/9 of the way towards (x1,y1) they will be at (3,1). What are the points (x1,y1) and (x2,y2)? b) Check your answer.

11) Prove Angle is Right Angle
ABC is an equilateral triangle. D is a point on segment AC twice as close to A as to C. E is a point on segment BC twice as close to C as to B. F is the point of intersection of the segments AE and BD. Prove: angle BFC is a right angle. [this is a good example of a proof where the coordinate geometry proof is straightforward but the classical geometry proof is quite difficult and very instructive to follow. I say to follow, because most people would not be able to do it on their own, but it is quite instructive to read and study the proof].

3.12) Make use of the 2d distance formula to prove that the 2d midpoint formula gives the point that it claims to give. (see MPF definition)

3.13) Use the 2d distance formula to prove that the 2d proportional point formula gives the point it claims to give. (see PPF definition)

3.14) Propose an equation for each of the following, then verify these proposals are correct by making use of the 3d distance formula. a) the 3d midpoint formula.
b) the 3d proportional point formula.

15) Find the midpoint of the segment (3,5,0) (1,1,-8) using
a) the 3d midpoint formula; b) the 3d proportional point formula.

3.16) Prove that the four diagonals of an arbitrary rectangular solid are congruent and intersect at a common midpoint.

17) (-1,0,7) is one end point of a segment. The trisection point of the segment closest to the other end point is (2,-5,6). Make use of the 3d proportional point formula to find the other end point of the segment.

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6 Locus Introduction
Line, Parabola, Circle and Locus Problems

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Locus problems are introduced in this section. In a locus problem, a shape (such as a circle) is defined geometrically, the task is then to find an equation for this set of points. In addition to locus problems. Several other interesting problems are given that require thinking and problem solving.

Lets re-introduce the concept of a general point. A general point represents more than one point. (x,2), (3,y), (x,y) are examples of general points. Since x is a variable which can represent any real number, likewise with y, the general point (x,y) represents all the points on the Cartesian plane. $y=2x$ is the equation of a line. We desire to construct a general point that represents all the points of this line. Into the place where the x coordinate goes, we put x and since $y=2x$, into the place where the y coordinate goes we put 2x. Doing this gives us the general point (x,2x) which represents all the points of this line. The various individual points of this line are gotten by substituting different values into x. If 0 is substituted into x, we get the point (0,0). For example if 1 is substituted into x, we get the point (1,2). Both of these points are points of the line $y=2x$.

Locus Example: Find the equation of the set of all points that are the distance 1 from the origin. We start with the point (x,y) and then set its distance from the origin equal to 1.

distance from (x,y) to (0,0) = 1 -> ... applying distance formula
$(x-0)^2 + (y-0)^2 = 1^2$ ->
$x^2 + y^2 = 1$ <---- the answer we seek, this is the equation of the set of all points that are a distance of 1 from the origin. This is the equation of the circle whose center is the origin and whose radius is 1.

More Example Problems
The problems of this section that have solutions in the back of the book are .. 1,13,20,21. These problem may serve as example problems if needed. Otherwise the student may do these problems on their own. The fact that these problems have solutions in the back of the book is made evident by the numbering format used. These problem are numbered as 4.1, 4.13, 4.20 4.21. Please read the Preface in the front of the book to learn more about problems which have solutions in the back of the book and how to easily access these answers.

Circles; Line, Parabola and Locus Problem Set

4.1) Locus Problem - Derive Equations of Circles
  a) Make use of the distance formula to find the equation of the set of all points that are a distance 5 from the point (2,3).
  b) Make use of the distance formula to derive the equation of the set of all points that are a distance r from the point (x1,y1).
  c) Can you see how the shifting theorem (see section 1) is in effect in problems a and b? Explain.

2*) What are the equations of the following circles? (don't calculate, do by inspection). a) center = origin, radius = 1; b) center = (-1,4), radius = 3.

3*) What are the center and radius of the following circles? a) x^2+y^2=5; b) (x-1)^2+(y+2)^2=4; c) x^2-2x+y^2+4y-11=0; d) x^2+y^2=6x+8y.

4*) What is the equation of the circle where one of its diameters is the segment whose end points are (5,-7) and (3,-1).

5) Find the equation of the circle in the first quadrant which is tangent to both axes and whose radius is 2.

6*) a) C(3,1) is the center of a circle. P(-1,4) is on the circumference of the circle. P(4,??) is an other point on the circumference of the circle. Find ?? b) C(x0,y0) is the center of a circle. P(x1,y1) is on the circumference of the circle. P(x2,??) is an other point on the circumference of the circle. Find ??.

7) Find the intersection of the line that intersects the y axis at -5 and the x axis at 2 .. and the circle with center (7,-2) and radius 9.

8) Find the intersection of the parabola y=7-x^2, and the line passing through the points (-3,-2) and (7,1).

9) Where does the circle centered at (-3,2) with radius 7, intersect the circle x^2-12x+y^2+8y+36=0?

10) What is the equation of the line perpendicular to 3x-7y+11=0 that passes through this line at y=-2?

11) a) What is the point on the line y=x+3 closest to the center of the circle x^2-10x+y^2-4x+25=0? b) What is the point on this line closest to this circle? c) What is the point on this circle closest to this line? d) How far apart are this circle and this line?

12) a) What is the distance between the circles $x^2-2x+y^2-6y+5=0$ and $(x+2)^2+(y+5)^2=9$?; b) What is the point on each of these circles that is closest to the other circle?

4.13) a) What is the equation of the circle centered at (5,1) and tangent to the line $6y+x=2$? b) Do this problem again using a different method.

14) A circle of radius 1 rolls left along the line $y=0$ until it makes contact with the line $y=(1/5)x$, at which time it comes to rest. Once the circle comes to rest, what is its equation?

15) Find the equations of the lines through the following point and tangent to the following circle. a) (0,13), $x^2+y^2=25$; b) (-8,3), $x^2-14x+y^2+10y+10=0$.

16) a) Show that the two circles $x^2+8x+y^2+6y=0$ and $x^2-16x+y^2-12y=0$ are externally tangent. b) Find the equation of the line tangent to both circles.

17*) a) Any three non collinear points define a circle. If a circle passes through the points (0,2), (6,9), (3,-4), a) what is the center of this circle? b) what is the radius of this circle? c) What is the equation of this circle? d) Do this problem again using a different method.

18*) a) Find the equations of all circles that intersects the points (1,4),(2,3) and have a radius of 5. b) Do this problem again using a different method.

19) On the circle $x^2+y^2=1$, a) find 4 points that are the vertices of a square; b) find 3 points that are vertices of an equilateral triangle. c) Do problems 'a' and 'b' again using a different method.

4.20) Locus Problem - Perpendicular Bisector
a) Make use of the distance formula to find the equation of the set of all points that are equidistant from the two points (1,2) (-1,7). b) Calculate the perpendicular bisector of the segment (1,2) (-1,7). c) Answers a and b should be the same? why?

4.21) Locus Problem - Parabola
a) Find the equation of the set of all points that are equidistant from the line $y=-1$ and the point (0,1). b) Find the equation of the set of all points equidistant from the line $y=-k$ and the point (0,k). c) What are the vertex and the axis of these parabolas?

22) A parabola whose axis is parallel to the y axis passes through the points (-3,3) (-1,1) (2,8). What is the equation of this parabola? Hint: The equation of any such parabola has the form Ax^2+Bx+C=0. (This was proven in section 2)

23) A bridge is to be built that requires an arch spanning 37 feet (horizontally) and is 11 feet high. Before the arch can be fabricated it is necessary to derive an equation defining its shape. Derive an equation for the arch, assuming it is part of a .. a) parabola; b) circle

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7 Line Problems

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In this section, students solve various problems by making use of lines and their equations. Using lines to solve all these problems (as opposed to other techniques such as Classical Geometry will help prepare the student to do coordinate geometry proofs.

<u>Line Problems Problem Set</u>

5.1) A man stands 24m from the base of a flag pole, the slope from the man's feet to the top of the flag pole is 1.23, a) How high is the top of the top of the flag pole? b) How far are the man's feet from the top of the flag pole?

2) An observer in a lighthouse 350 feet above sea level observes two ships due east. The slopes of depression to the ships are 0.0700 and 0.1140. a) How far apart are the ships? b) How far from the observer to each of the ships?

3) An airplane flying horizontally at a constant heading at 560 mph is spotted by an observer at an slope of 0.601 from the horizontal. One minute later it is directly overhead. How high is the airplane?

4) From physics it is known that if an arrow shot at any given speed S, it will go further if shot at a slope of 1 with respect to the horizontal and vertical than if it is shot at any other slope. If an arrow is shot at a speed of 200 ft/second, what will be its vertical and horizontal speed immediately after it is shot if a) it is shot at a slope of 1? b) it is shot at a slope of 1/2? c) it is shot at a slope of m?

5) Find the slope of the angle bisector of the angle formed by the following two rays. Both rays have a common end point of the origin (0,0) and exist in the in the 1st quadrant. One ray has a slope of 5 the other ray has a slope of 7.

6) A airplane flies to a city 236 miles away. After flying 157 miles "towards" that city, the pilot realizes he has been flying a little off course at a heading of slope 0.2000 instead of at the proper heading of slope 0.0000. a) What heading does the plane now have to fly at in order to reach its destination?; b) If it is assumed that the current mistaken heading has a slope of 0, at what slope does the plane have to fly at in order to reach its destination?

7) A(-3,2) and B(5,12) are two of the vertices of triangle ABC. A line through G, the midpoint of AB, and parallel to AC, intersects BC at H(10,2). Find the coordinates of C, the third vertex.

8) Ray 1 has a slope of 1, ray 2 has a slope of 2. Both rays have their end point at the origin and are in the first quadrant. In an alternate (rotated) coordinate system where both ray's end point are still at the origin, and ray 1 lies along the positive x axis, what is the slope of ray 2?

9) From the sunroof of an apartment building, the slope of depression to the base of an office building is 1.26 and the slope of elevation to the top of the office building is 0.939. If the office building is 847 ft high, how far apart are the two buildings and how high is the apartment building?

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8 Proving Theorems, Dot Product and Locus Problems

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Most of this section focuses on proving Geometry problems by making use of Coordinate Geometry. If you started this book at the beginning and have done all or most the problems up to this point, you already have experience doing several Algebra and Coordinate Geometry proofs.

This section starts out with proofs that are quite easy, from there the proofs get more challenging and more interesting. The first rather challenging proof in this section is problem 6.7 Prove: The diagonals of all rhombuses are perpendicular. Do all the proofs in this section that lead up to this problem, then try to do this problem on your own without looking at the solution in the back of the book. If you can do this you are well on your way. If not, if after struggling to do this problem you find you are stuck. Follow the general rule of study of this book and look at the solution in the back of the book, just enough to get the needed hint(s). After that try to do the problem again all by yourself, without looking.

Example Problems
The problem(s) of this section that have solutions in the back of the book are .. 6,7,8,28,42. These problem(s) may serve as example problems if needed. Otherwise the student may do these problems on their own. The fact that these problems have solutions in the back of the book is made evident by the numbering format used. These problem are numbered as 6.6), 6.7), 6.8), 6.28), 6.42). Please read the Preface in the front of the book to learn more about problems which have solutions in the back of the book and how to easily access these answers.

Proving Theorems; Dot Product; Locus Proofs Problem Set

1) Prove: The quadrilateral (0,0)(1,0)(1,1)(0,1) is a square.

2) Prove: The quadrilateral (1,2)(sqrt{3}+1,3)(sqrt{3}(3+sqrt{3}) (0,2+sqrt(3)) is a square.

3) Prove: The triangle (4,-1)(5,6)(1,3) is an isosceles right triangle.

4) a) Prove the diagonals of the rectangle, (0,0) (a,0) (a,b) (0,b) are equal in length. b) Does proving this, prove that the diagonals of all rectangles are equal in length? why? why not?

5) Prove: The diagonals of all squares are perpendicular.

6.6) a) Prove: The diagonals of all parallelograms bisect each other. b) Solve this problem using a different method.

6.7) Prove: The diagonals of all rhombuses are perpendicular.

6.8) Prove: If the diagonals of any quadrilateral are perpendicular and bisect each other, then the quadrilateral is a rhombus.

9) Prove: The midpoint of the hypotenuse of any right triangle is equidistant from the three triangle vertices.

10) S is an arbitrary square. S has a point on each of its four sides. Each of these points is a distance d clockwise from a vertex of S. Prove that these four points are themselves vertexes of a square.

11) Prove: The medians to the congruent legs of any isosceles triangle are congruent.

12*) Prove the Midline Theorem: The segment joining the midpoints of two sides of any triangle is parallel to the third side and half as long.

13) Prove: The segment joining the midpoints of the diagonals of any trapezoid is parallel to the bases and its length is one-half the difference of the lengths of the bases.

14) Prove: a) The midpoints of the sides of any triangle T, are vertices of a triangle that is similar to T. b) What is the ratio of the triangle areas?

15) Prove: The midpoints of the sides of any isosceles trapezoid are vertices of a rhombus.

16) a) Prove: The midpoints of the sides of any quadrilateral are vertices of a parallelogram. b) (Optional) Prove: The area of this parallelogram is half the area of the quadrilateral.

17) Prove: If the diagonals of any quadrilateral bisect each other, then that quadrilateral is a parallelogram.

18) Prove: In all Parallelograms, the sum of all -sides squared- equals the sum of all -diagonals squared-.

19) Prove: In all quadrilaterals the sum of all -sides squared- is equal to the sum of all -diagonals squared- plus four times the square of the length of the segment between the midpoints of the diagonals.

20) An arbitrary square has 3 segments in its interior. S1 is from the lower left corner to the upper right corner. S2 is from lower right corner to the midpoint of the left side. S3 is from the midpoint of the right side to the upper left corner. Prove: S2 and S3 trisect S1.

21) An arbitrary square has 3 segments in its interior. S1 is from the lower left corner to the upper right corner. S2 is from lower right corner to the midpoint of the left side. S3 is from the lower right corner to the midpoint of the top side. Prove: S2 and S3 trisect S1.

22*) Prove: If the vertex of an angle is on a circle and the rays of this angle pass through both endpoints of a diameter of the circle then this angle is a right angle.

23) A and B are end points of a diameter of a circle. p is a point on this diameter, (p is not A or B). A line passes through p and is perpendicular to segment AB. q is a point on this line, q is also on the circle. Prove Ap * pB = (pq)^2.

24) a) If two sides a and b of an arbitrary triangle have slopes m1 and m2 where m1*m2=-1, prove that a^2+b^2=c^2 where c is the third side. b) Assuming neither a or b is aligned with the x or y axes, prove the converse. (In these proofs, do not make use of the fact that if m1*m2=-1, that these slopes are perpendicular, in other words, do not make use of the fact that the angle opposite c is a right angle).

25) a) P is any point in the plane of an arbitrary rectangle. A,B,C,D are the vertices of the rectangle in clockwise order. Prove: (PA)^2+(PC)^2=(PB)^2+(PD)^2. b) Make use of this theorem to prove the Pythagorean theorem.

26) Prove: In any triangle ABC, if segment CM is the median to AB, then AC^2+BC^2=1/2AB^2+2CM^2.

27*) Two circles of different size are internally tangent at a point P. The end points of a diameter of the smaller circle are, the center of the larger circle and the point P. Prove that any segment from P to any other point on the larger circle is bisected by the smaller circle.

6.28) Prove: (Vertical lines excepted) Two lines are parallel if and only if they have the same slope.

29*) Dot Product (2d)
The dot product of two points (x1,y1) and (x2,y2) is defined as x1*x2 + y1*y2. a) Prove: If the dot product of any two points (x1,y1) and (x2,y2) is zero, then the lines (0,0)(x1,y1) and (0,0)(x2,y2) are perpendicular. b) Prove the converse.

30) Dot Product (3d)
The dot product of two points (x1,y1,z1) and (x2,y2,z2) is defined as x1*x2 + y1*y2 + z1*z2. a) Prove: If the dot product of any two points (x1,y1,z1) and (x2,y2,z2) is zero, then the lines (0,0,0)(x1,y1,z1) and (0,0,0)(x2,y2,z2) are perpendicular.
b) Prove the converse.

31) Make use of the dot product to prove the following angles are right angles, then make use of the Pythagorean theorem to determine if your answers are correct. a) (8,5)(0,0)(-5,8);
b) (1,-2,7)(0,0,0)(-13,4,3).

32) Make use of the dot product to determine if the following triangles are right triangles and then make use of the Pythagorean theorem to verify that your answers are correct.
a)(2,17)(15,20)(10,12); b)(-2,3,1)(11,-1,-2)(13,-3,5).

33) Make use of the dot product to come up with a right triangle such that; the coordinates of its vertices are integers, none of its vertices are located at the origin, none of its sides are parallel to any of the axes. Then make use of the Pythagorean theorem to verify you have done this correctly, that your triangle is a right triangle. Do all of this in a) two dimensions;
b) three dimensions.

34) a) Consider the angle (-1,7)(0,0)(a point on the line y=1/2*x-5). Make use of the dot product to find the point on this line so that this angle will be a right angle. Make use of the Pythagorean theorem to verify your answer is correct. b) Do this again for the angle (4,-4)(2,-9)(a point on the line y=-2x+11).

Note: The dot product is a useful concept in physics, where it is usually defined in terms of vectors. If points used in this definition were interpreted as vectors, the definitions would be similar.

35*) Locus Problem - Prove that the set of points equidistant from any line and any point (not on that line), is a parabola. [This point is referred to as the focus of the parabola, this line is referred to as the directrix of the parabola, see problem 46 in the Challenging Problems section to learn more about the focus and directrix].

36) Locus Problem - Prove that the set of points equidistant from any given point is a circle. (Yes, this is the common definition of a circle, however this is still a legitimate problem. The student must demonstrate, that an object defined this way has an equation that is (now) known to be the equation of a circle).

37) Locus Problem - Prove that the set of points that are equidistant from any two points is a line. Bonus: Show that this line is the perpendicular bisector of the segment defined by the two points.

38*) Locus Problem - Prove: The set of points that are c (c>0) times as close to one point as to another point, is a circle.

39) Locus Problem – P is a point on a line L, prove that the set of points whose -distance to P- squared, equals their distance to L, is a circle. (Actually it is two circles). Bonus: Graph both circles.

40) Locus Problem - A and B are two perpendicular lines which intersect at a point Q. P is a point such that its distance to Q squared equals its distance to A plus its distance to B. Prove that the set of all such points P is a set of four circles. Bonus: Graph these circles.

Note: The geometric mean of two positive numbers a and b is sqrt(a*b).

41) Locus Problem - A point lies inside an isosceles right triangle. The distance from this point to the hypotenuse equals the geometric mean of .. the distance from this point to one leg of this triangle, AND the distance from this point to the other leg of this triangle. Prove that the set of all such points is a quarter of a circle.

6.42*) Locus Problem - There is an alternate proportional point formula (PPF) which specifies the coordinate(s) of the point that divides a segment into 2 parts, such that the ratio of the lengths of the two resulting segments is expressed as a constant. a) Given a segment AB, in one dimension, [Assume the segment AB lies on the number line and that the location of each of its endpoints are specified by a single number]. Using the locus technique derive a PPF that finds a point p, such that Ap divided by pB is the constant c. b) Make use of this PPF to derive the standard PPF previously derived in this book, i.e. p=A+c(B-A)(this PPF finds the point p that divides a segment into two parts, such that the ratio between the segments Ap and the entire segment is a given constant]. c) Now to practice what has been learned, use the locus technique to derive the standard PPF, then make use of the standard PPF to derive the alternate PPF.

*** This is the dividing line between regular and Advanced Coordinate Geometry. ***

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9 Constraints

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A point that is constrained can not exist everywhere, there are places where it is not allowed to be. A shape that is constrained has limits placed on it. For example the size of a circle may be restricted. A quadrilateral may be restricted to being a rectangle with whose sides have certain ratios.

In this section, unless circles or the use of circles is specifically mentioned, do not make use of circles to do these problems.

Constraints Problem Set

1) Distance: Point(s) on Line - Point
A line L contains the point (1,5) and has a slope of 7. What point(s) on L are a distance 12 from the point (-1,2).

2) Distance: Point on Parabola - Point
What are the points on the parabola $y=2x^2+x+5$ that are a distance 12 from the point (-2,1)?

3) Distance: Points on Parabola - Line
What are the points on the parabola $y=x^2$ that are a distance of 7 from the line with a slope of 2 and a y intercept of -6?

4) Distance: Points on Parabola - Circle
What are the points on the parabola $y=x^2+1$ that are a distance of 7 from the circle $x^2-6x+y^2+2y+9=0$?

5) What is an equation of the general line that has a) any slope and passes through any point? b) any slope and has any y intercept? c) Verify that the equations of the lines in a) and b) are equivalent.

6*) What is the equation of the family of lines with the following sets of constraints? a) The slope of the line equals its y intercept; b) The slope of the line equals its x intercept; c) The slope of the line is the x coordinate where it intersects the line $2y-x-2=0$.

7) a) What is the equation of a the general circle a) with any center and any radius? b) with its center on the line $y=3x-2$.

8) a) What is the equation of the general circle that contains the point (a,b)? Make a diligent effort to do this, if you can't, then do the following lead up problems, and then do this problem. STOP b) What is the general equation of the circle whose radius equals the distance from its center to the origin? c) What is the general equation of the circle that contains the point (0,0)? d) What is the general equation of the circle that contains the point (a,b)? e) There is an other approach that leads to the same answer, can you think of it? Do it.

9) What is the equation of the general circle with the following constraints, a) center is on the line y=x, and tangent to x and y axis. b) center on line y=2x+3 and tangent to the x axis? c) tangent to lines y=x and y=2x+3?

10) An equilateral triangle is inscribed in a unit circle centered at the origin. Find one possible set of coordinates for the vertices of such a triangle. b) Do this problem again, using a different approach. (A unit circle is a circle whose radius is one).

11*) a) Make use of the distance formula to find the midpoint of the segment (-3,4)(5,-4)? b) Make use of the distance formula to find the trisection point of the segment (-3,4)(5,-4) closest to the point (5,-4).

7.12) Alternate Proportional Point Formula (PPF) - Derivation
a) Given two points P1(x1,y1) and P2(x2,y2) let p be the point on the segment P1 P2 (or the line P1 P2) such that (the length of the segment) P1 p divided by P2 p equals c. Calculate the coordinates of c. b) Make use of this alternate form of the PPF to derive the more standard form of the PPF given in this book previously i.e. where p is the point such that c equals P1_p divided by P1_P2. This proportional point formula in vector form is p=P1+c(P2-P1).

13) Calculate what constraints on an isosceles triangle are necessary, so that the medians to the congruent sides of the triangle will be perpendicular. Hint: One possible way to define the constraint(s) would be as the ratio of base to height.

14) ABC is a right angled triangle, its point C lies on the line y=3x. A is (2,1) and B is (5,5). Find all possible coordinates of C.

15*) (Make use of the distance formula to do this problem) a) If you start at point (-4,1) and travel a distance of 3 at a slope of 2, in a left downward direction, where will you end up? b) If you start at point (1,1) and travels a distance of 4 towards the point (-7,-2) where will you end up?

16*) Derive an equation for the family of circles that contain the point (-1,-2) and are tangent to the line y=4-3x. (A similar problem is in the Challenging Problems section to benefit students who will not do both sections).

17) In the "Proving Theorems Using Coordinate Geometry" section it was proven that the midpoints of the sides of any quadrilateral are vertices of a parallelogram. Determine what constraint(s) on a quadrilateral are required in order for this parallelogram to be a) a rectangle; b) a rhombus; c) a square.

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10 Parametric Equations

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Important: Before doing any problems in this book, read those parts of the Preface which are underlined and highlighted in red.

Parametric equations are an other way to express functions. y=3x+2 is the equation of a line in classical (non parametric) form. This line can be expressed in parametric form by the two equations x=t and y=3t+2, [this is standard (parametric) form], t is the parameter of this set of parametric equations. Parametric form of a function, specifies the location of each of the coordinates, the x coordinate and the y coordinate - independently. Any particular point of this line is realized by substituting an appropriate value into t. Other ways to express this line parametrically are (t,3t+2), [this is general (parametric) form] or as (0,2)+t(1,3) [this is point slope (parametric) form]. Parametric form of a general function can be written as the pair of equations x=X(t) and y=Y(t) where X(t) and Y(t) are functions of t. The general function above can also be represented parametrically as (X{t},Y{t}). For example, a parabola could be represented parametrically as (t,t^2).

I) Express the line y=1-2x in a) standard (parametric) form; b) general (parametric) form; c) point slope (parametric) form.

II) Given the line (t,4+3t), give 2 points that are on this line.

III) a) Express a general function in parametric form.

IV) a) In the three problems above, what are the parameter(s) you used to express each of the parametric equations? b) When expressing an equation in parametric form, the x and y coordinates are each defined independently in terms of a third variable, the parameter. How does this differ from equations expressed in classical form.

In 2 dimensions, slope of a line is given by the fraction, (change of y)/(change of x). In 3 dimensions slope (orientation) is given by a ratio, (change of x: change of y: change of z). Slope in two dimensions may also be given by a ratio, (change of x: change of y). In two dimensions a slope of 2/5 can be denoted in ratio form as (5:2). In general, a slope of (change of y)/(change of x) is denoted in ratio form as (change of x:change of y).

V) a) Express the slope 5/8 parametrically; b) Express the slope (3,-2) in classical form; c) Give an example of a slope in 3 dimensions; d) What is another word for slope? In three dimensions, is this word a better word than slope? Why or why not?

In two dimensions a line is denoted by an equation involving the variables x and y, i.e. y=2x+1. It is not possible to do something similar in three dimensions, y=3x-z+1 for example is a plane not a line. To denote a line in 3 dimensions, parametric equations are used. The parametric equation of a 3d line with slope (1:2:3) passing through the origin could be denoted as (t,2t,3t) or as t(1,2,3), "When" t=0, the corresponding point on this line is the origin, "when" t=2 the corresponding point on this line would be (2,4,6). The 3d line passing through the point (-1,5,7) with slope (3:1:-2) could be expressed in parametric form as (-1,5,7)+t(3,1,-2) also as (-1+3t,5+t,7-2t) or as x=-1+3t,y=5+t,z=7-2t. In 2 dimensions, the line passing through the point (1,-7) with a slope of 2/3 could have any of the parametric forms. (1,-7)+t(3,2) or x=1+3t,y=-7+2t or (1+3t,-7+2t).

Parametric Line Form Examples
Point Slope (parametric): (-1,5,7) + t(3,1,-2)
Standard (parametric): x=-1+3t; y=5+t; z=7-2t
General (parametric): (-1+3t,5+t,7-2t)

VI) a) Why are parametric equations used to express lines in 3 dimensions? b) Without looking, write the names of the 3 parametric line forms and beside each of these names, write the line y=(2/3)x-6 in that corresponding line form.

Parametric Equations Problem Set

1) A particular line may be expressed in parametric form as either (0,4)+t(1,-2) or as (t,4-2t). What point on this line is represented when t equals a) 0 ?; b) 1 ?; c) -5 ?.

2) a) Express the line x=t; y=3t+7 in general (parametric) form.
   b) Express the line (-7,1)+t(5,-2) in standard (parametric) form.
   c) Express the line (-1,3)+t(5,2) in general (parametric) form.
   d) Express the line (2-t,4t-7) in point slope (parametric) form.
   e) Express the line (-12,0,7)+t(1,-2,4) in general (parametric) form.
   f) Express the line (11t-2,3,5+6t) in point slope (parametric) form.

3) Express the line passing through point (2,-3) with slope of 3/8 in a) point slope form; b) a parametric form.

4) Given the line (-9,17,5)+t(11,-8,19). What value of t is required to realize the point a) (-152,121,-242); b) (68,-39,138).

5) Express the line with x intercept of 8 and y intercept of -3 in a) intersept intersept form; b) point slope form; c) in parametric form.

6) Express the following in parametric form. a) y=mx+b; b) ax+by+c=0.

7) What is the slope in parametric form of the line passing through the following point pairs? a) (0,0) (2,3); b) (-1,2) (7,4); c) (2,-1,7) (-2,1,4).

8) Assume that t is time (in seconds) and each of the following parametric expressions denotes a point in motion, I(t,1-2t) and II(t,3t-5). For each of these lines a) graph this line; b) mark the points on this line where t=0 and where t=1; c) Draw an arrow on the line indicating the direction of travel of the parameterized point; d) Calculate how fast the parameterized point is moving. (Assume the distance between the origin and the point (1,0) is 1 unit).

9) Express line (a,b)+t(c,d) in a) point slope form; b) general form.

10) What is a parametric equation of the line with slope (3,-2,1) passing through the point (-1,1,2)?

11*) Express the line passing through the points (-3,0,7)(2,1,5) in a) point slope (parametric) form; b) What is another obvious point slope (parametric) expression of this line?

12*) Express the line x=12-7t; y=t+5 a) in general (parametric) form; b) In slope intercept form; c) in point slope (parametric) form; Note: keep your work so you can compare your answer with problem 15 a.

13*) Express the line (7-3t,8t+4,1-2t) in a different way, such that the x coordinate is not denoted by 7-3t, but by a) t; b) 2t+5.

14) Do a and b making use of the technique learned when doing problem 13. a) Do problem 12 b again; b) Convert the line (7-2x,3x+1) to general form.

15) Prove that the following line pairs are the same line.
a) (5,12)+t(-1,5) & (10,-13)+t(-1,5);
b) (-7,17,-5)+t(2,-5,3) & (9,-23,19)+t(2,-5,3)

16) Prove that the following point triples are collinear.
a) (1,19) (3,14) (21,-31); b) (-8,3,-5) (17,-7,15) (27,-11,23)
c) Do a and b again using an alternate method.

17) a) Consider the angle (0,9,-2)(-1,2,3)(a point on the line {2-t,10+2t,14-3t}). Make use of the dot product to find the point on this line so this angle will be a right angle. b) Make use of the Pythagorean theorem to verify your answer is correct.

18*) Find where each of the following lines intersect and then check each answer and ensure it is correct before moving on. Do not convert any line to classical form to solve, stay within parametric form.
a) (t,1-3t)(t,7t-5); b) (t-2,5-3t)(7t+1,t-14);
c) (54-13t,t-5,8t-27)(5t+37,20+3t,-86-13t)

19*) Is it possible to have a four sided figure (tetrahedron) in space, where all four sides are congruent equilateral triangles? a) Make a convincing argument that such a shape exists. b) Find one possible set of coordinates for the vertices of such a figure. c) Draw this shape. d) Determine if the altitudes of this tetrahedron are concurrent, (meaning do all altitudes of the tetrahedron intersect at a common point)? e) Calculate the slope of the angle between one pair of adjoining faces. f) Prove: If all faces of a tetrahedron are equilateral triangles, then all of its faces are congruent. (This proof isn't necessarily a coordinate geometry proof).

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11 Vectors

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Important: Before doing any problems in this book, read those parts of the Preface which are underlined and highlighted in red.

Other sections of this book do not make much use of vectors, preferring instead to solve problems as close to the classical way as possible, which gives the student more opportunity to learn and practice the skills of problem solving. (Vector techniques make some problems easier to solve). It is a great advantage however with no down side to define the proportional point formula (PPF) and midpoint formula (MPF) in terms of vectors. Doing this allows a more intuitive derivation of these formulas and also makes their use more intuitive. The proportional point formula is an important fundamental theorem of coordinate geometry. Defining it in vector terms provides economy of notation, which helps to make some proofs and solutions to problem easier to visualize and understand. The derivation of the proportional point formula in this section will be an intuitive derivation, not a rigorous one. In the "Midpoint Formula & Proportional Point Formula" the properties of the proportional point formula are proved in a more rigorous manner.

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Vector Introductory Discussion Topics

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A vector is something (a mathematical object) that has a magnitude and a direction. Velocity is an example of a vector. Speed is one of the components of velocity and direction is the other. Speed is the magnitude of velocity, and unlike speed, velocity has a direction. A quantity such as speed, or even a regular number such as 5 which has a magnitude but no direction is referred to as a scalar.

I) State whether the following are vectors or scalars and give the reason(s) why. a) velocity; b) temperature; c) position; d) speed

On the coordinate plane every point is associated with a vector, how can this be? What is the magnitude and what is the direction of a point on a plane? A vector would seem to require two points, the direction in going from one point to the other point would be the direction of the vector, and the distance from one point to the other would be the magnitude of the vector, so how could one point represent a vector? This is how, the other point associated with each point (or vector) on a plane is the origin or (0,0). So the vector <1,1> would then have a magnitude that is the distance between (0,0) and (1,1) or sqrt(2). The direction defined by these two points is the direction one goes when going from the origin (0,0) to the point (1,1). This direction is considered to be 45 degrees*, which is the angle from the positive x axis to .. the ray from (0,0) to (1,1). An angle counter clockwise from the positive x axis is considered to be positive, an angle clockwise from the positive x axis is considered to be negative. Unlike in Geometry, an angle may be greater than 180 or even 360 degrees or less than -180 or -360 degrees.

II) Since in principal it would take two points to define a vector, how is it we can identify a vector "using only one point"?

III) What are the magnitudes (distances) and directions (angles) associated with each of the following vectors? a) (1,1); b) (2,sqrt{3}); c) (-1,-sqrt{3}/2);

The point (0,0) is also considered to be associated with a vector, the null vector, this vector has no direction (direction is undefined) and its magnitude is 0.

IV) All vectors have magnitude and a direction, this is the rule. The
null vector has 0 magnitude and doesn't have any direction, this is an _____________ to the rule.

Any vector on the x-y plane may be expressed in two forms. The vector <1,1> is expressed in Cartesian form. This same vector may also be expressed in polar form as <sqrt(2),45deg>, (deg means degrees) here sqrt(2) is the distance from the origin to the point (1,1) and 45 degrees is the angle from the positive x axis to the ray with endpoint (0,0) passing through the point (1,1).

V) State in which form each the following vectors are expressed, Cartesian or polar, and then write each vector in its alternate form. a) (1,0); b) (0,1); c) (3,60deg); d) (8,120deg); e) (0,-1); f) (-2,2)

Vectors have a head and a tail, in the vector <1,1>, the point (1,1) is the head and the point (0,0) is the tail. This is because the direction of the vector goes from (0,0) towards (1,1). In a vector, the direction of the vector goes from the tail towards the head.

Vectors are not necessarily fixed on the coordinate plane. Although one point (1,1) or two points such as (0,0) and (1,1) define a vector, the vector does not necessarily lie at this location between these two points. These two points merely establish a magnitude and a direction and this vector may lie anywhere on the plane. A vector whose tail lies at the origin is said to be in “home position”. Vectors are frequently denoted by an arrow, i.e. -----> with the arrow representing the head part of the vector and the other side of the arrow representing the tail.

VI) What is the head and tail of each of the following vectors?
a) (1,1); b) (0,-1); c) (4,60deg); Hint: A head may be expressed in either polar or Cartesian format.

Vectors may be added to each other, here is an example, <1,1>+<2,1>=<3,2>, following is the rule for adding vectors in Cartesian form, v1<x1,y1>+v2<x2,y2>=v3<x1+x2,y1+y2>.

VII) a) (1,-2)+(3,7)= ?; b) (-7,-2)+(-1,4)= ?

Vectors may also be added graphically. To add two vectors v1 and v2, graphically, i.e. v1+v2 one would place the tail of v2 on the head of v1. The tail of the resultant vector would be at the tail of v1, (not necessarily the origin). The head of the resultant vector would be at the head of v2. Since vector addition is commutative it makes no difference whether v1+v2 or v2+v1 is done.

VIII) Using graph paper, graphically add/subtract the vectors you added analytically in the previous problem, verify the answers are the same.

Vectors may be multiplied by numbers. A vector multiplied by 3 becomes three times as long but points in the same direction. If a vector is multiplied by zero, it becomes the null vector, without direction and having zero magnitude. If a vector is multiplied by a negative number, -2 for example, it becomes twice as long but then points in the opposite direction. Where c is a number, the following equations illustrate a vector being multiplied by a number, or in vector speak by a scalar.

c*<x1,y1>=<c*x1,c*v2> or 3<1,2>=<3,6> ... Cartesian form
c*<r,o>=<c*r,o> or 7<2,45deg>=<14,45deg> ... polar form

also (where V is a vector)

-<x1,y1> = -1*<x1,y1> = <-x1,-y1> or -<1,2> = -1<1,2> <-1,-2>
-V=-1*V

Note: A vector that has a magnitude of 1 and is pointing at an angle of 45 degrees, may be expressed as a vector having a magnitude of -1, and having an angle of 45deg + 180 deg = 225deg. so <1,45deg>=<-1,225deg>. In general <r,o deg>=<-r,o + 180 deg>

A vector may also be represented by a single variable, A=<3,4>= A<3,4> for example.

Definition - Vector Subtraction: Where A and B are vector, A-B means A+(-B)

IX) Where V=(2,-1) and N=(4,5) calculate the following. a) 2V; b) -N; c) N+3V; d) 2V-N; e) -V+2N

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Vector Properties and Definitions
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Before reviewing the following 12 vector properties, student should review the field properties of (real) numbers, to familiarize themselves with these types of properties with a mathematical object they are already familiar with (numbers).

Students should be familiar with the following 12 vector properties and take the time to convince themselves that each of these vector properties are true.

In the following section, small letters represent an arbitrary scalar (real number), and capital letters represent an arbitrary vector.

1) A+B is a vector; closure addition
2) cA is a vector; closure scalar multiplication
3) 0A is the null vector; null vector
4) A+(-A)=(null vector); any vector added to its negative self is the null vector
5) A+(null vector)=A; null vector addition
6) A+B+C=A+(B+C)=(A+B)+C; associative property of vector addition
7) A+B=B+A; commutative property of vector addition
8) c(A+B)=cA+cB; distributive property of vector addition
9) A-B=A+(-B); definition of vector subtraction
10) A+C=B+C implies that A=B; reversibility of vector addition
11) cA=cB implies that A=B; reversibility of scalar - vector multiplication
12) A+?=B implies that ?=B-A; for proof see end of this section

The above properties illustrate that vectors, when added together or when multiplied by a number have properties similar to numbers. A main difference between vector arithmetic and number arithmetic is that vectors can't be multiplied together, (Vector multiplication is defined but not here, and vector multiplication doesn't the same properties as multiplication of numbers).

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Vector Problems
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1) The null vector has 0 magnitude and no direction, why is it "necessary"?

2) When a vector is multiplied by -1, does this affect its magnitude? Does this affect its direction? explain.

3) Draw each of the following vectors, i.e. draw a segment from the tail of each vector, (the origin), to the head of each of the vectors, at the head of each vector, put a small arrow. a) <-1,7>; b) v2<4,2>; c) -v3<3,-5>; d) -2*<4,2>

4) Where v1=<7,-8>; v2=<-1,3>; v3=<1,1>, do the following analytically and graphically, compare answers from each method. a) <-1,4>+<7,2>; b) <-5,0>-<1,2>; c) v1+v2; d) v3-v1; e) v2+v3;

6) Where v1=<-3,0>, v2=<1,1>, v3=<1,3>; simplify each of the following a) 2<1,0>; b) -<-1,0>; c) -v1+v2; d) v1+2*v2; e) –3<v1-v3>; f) 0(v2)

Hint: A vector in simplified form is not multiplied by a number and does not have a negative sign in front if it, which is the same as being multiplied by -1. The answer to each of these questions is a simplified vector.

7) a) All vectors are associated with a point. The null vector is associated with which point? b) <-2,9> is associated with what point? c) <2,17deg> is associated with what point?

8) Without looking at the vector properties above, (if possible), define vector subtraction.

9) Draw each of the following vectors.
a) <1,30deg>; b) <-1,30deg>; c) <-1,210=30+180deg>; d) <1,-30deg>;
e) <2,30deg>, f) 2<-3,45deg>, g) 0*<2,45deg>; h) –2<-1,45deg>
i) <2,720deg>; j) <2,-360deg>;

10) Convert the following vectors to Cartesian form
a) (1,30deg); b) (2,45deg)(5,120deg);
d) (-1,30deg); e) (-1,30deg+180deg);

11) Demonstrate that –(a,m deg) = (-a,m deg);

12) Demonstrate that (-a,m deg) = (a,m deg + 180deg);

13) Show that the laws of associativity and commutativity hold for vector addition, v1=<-1,4>, v2=<3,2> by demonstrating analytically and graphically that the following equations are true.
a) v1+v1=v2+v1; b) v1+(v2+V3)=(v1+v2)+v3

14) A is the vector with tail at (-1,2) and head at (3,1). B is the vector with tail at (5,2) and head at (3,7). C is the null vector. Represent each of the following vectors in Cartesian form. a) A; b) B; c) C; d)A+B; e) A-B; f) -A+C; g) -(A+B); h) 2A, i) -3(B-A)

15) Any vector not in home position, whose tail is at the point (x0,y0), whose head is at the point (x1,y1) is represented in home position as the vector (x1-x0,y1-y0). Show that this representation is correct by showing that it preserves the magnitude and direction of the vector.

16) The following sets of points are vertices of rhombus given in counter clockwise order, starting with the lower left vertex. Make use of vectors to determine all vertices. In the distance formula section of this book, the distance formula was used to solve this problem. Use of vectors makes solving these problems easier.
a) (0,0)(5,0)(8,4)(x,y);
b) (-1,4)(4,4)(7,8)(x,y);
b) (5,1)(6.9225,y2)(8.1261,3.1486)(x3,2.5973)
c) (2.1478,1.5975)(x2,2.5715)(5.5744,y3)(x4,3.9908)

17) a) Calculate the magnitude of (3,7)? b) Calculate the magnitude of 5(3,7). c) Prove: If a vector V has a magnitude of m, then the vector cV has a magnitude of cm.

Definition: Vector Normalization - A vector is normalized if its orientation remains the same, but its magnitude is changed to 1.

Definition: Vector Resizing - A vector is resized if its orientation remains the same, but its magnitude is changed to any other value.

18) Vector Normalization and Resizing
R is the vector (2,6). a) Without making use of the distance formula, determine the vector cR (c is a constant) that has a magnitude twice that of (2,6), then make use of the distance formula to check your answer. b) Without making use of the distance formula, determine the vector cR that has a magnitude half that of (2,6). c) Using the distance formula, determine the magnitude of (2,6) then using the same method used in a and b determine the vector cR that has a magnitude of 1. e) Determine the vector cR that has a magnitude of 7. f) V is the vector (a,b) and lies in the first quadrant. Determine the vector cV that has a magnitude of k.

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Vectors: Proportional Point Formula
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Consider the vectors A and B. We wish to find the vector which if added to A will equal B. In other words, A+?=B

rearranging by using vector arithmetic we get

?=B-A

For rigorous justification of this rearrangement see the last problem at the end the this section.

If someone starts at point A and wishes to go ALL THE WAY to the point B, what vector would they add to A? Answer is (B-A), i.e. A+(B-A) is B. It seems reasonable then, if someone starts at the point A and wants to go ..2/5.. of the distance towards B, they would add (2/5)(B-A) to A. The equation we propose for accomplishing this objective then is P=A+(2/5)(B-A). In the section "Midpoint Formula & Proportional Point Formula", it is formally proved this reasoning leads to a correct result.

We wish to make this equation more general.

If we are at point A and wish to go the fraction c (0<=c<=1) from A towards B along the segment AB, the following equation will give us the desired location or point.

P=A+c(B-A) Proportional Point Formula

1*) What vector do we need to add to the point (-1,3) to reach point (7,8)?

2*) Find the point that is the point that is 80 percent of the way from (0,1) towards (5,-2)?

3) If p1 is (-1,2) and p2 is (4,-5), what is the midpoint of the segment p1 p2?

4) Divide the segment (1,1) (2,5) into 3 equal segments. (Give the set of points that does this).

5) If you start at the point (-1,7) and go a distance of 2 towards the point (6,3), where will you be?

6) If you are at point A, and travel 2/13 of the way towards point B you will be at (-1,7). If you are at point A and travel 5/6 of the way towards point B you will be at point (4,11). What are points A and B?

7) Where A and B are vectors, prove that A+?=B implies ?=B-A. To do this make use of the table of vector properties given previously in this appendix. Try to do this yourself. If you are unable, review the proof below, then prove this theorem by yourself without looking. Perhaps after doing this proof you can find an even easier or a different way to do it.

```
1) A+?=B  given
2) (A+?)+(-A)=B+(-A) . .  reversibility of vector addition property.
3) (?+A)+(-A)=B+(-A) . .  commutative property of vectors.
4) ?+(A-A)=B+(-A)  . . .  associative property of vectors.
5) ?+null vector=B+(-A).  definition of null vector
6) ?=B+(-)A  . . . . . .  any vector added to the null vector is
                          the vector itself
7) ?=B-A . . . . . . . .  definition of vector subtraction
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Dot Product
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1*) Dot Product (2d)
The dot product of two vectors (x1,y1) and (x2,y2) is defined as x1*x2 + y1*y2. a) Prove: If the dot product of any two vectors in two dimensions is zero, then these vectors are perpendicular; b) Prove the converse.

2) Dot Product (3d)
The dot product of two vectors (x1,y1,z1) and (x2,y2,z2) is defined as x1*x2 + y1*y2 + z1*z2. a) Prove: If the dot product of any two vectors in three dimensions is zero, then these vectors are perpendicular; b) Prove the converse.

3) Make use of the dot product to prove the following angles are right angles. a) (8,5)(0,0)(-5,8); b) (1,-2,7)(0,0,0)(-13,4,3).

4) Make use of the dot product to determine if the following triangles are right triangles and then make use of the Pythagorean theorem to verify that your answers are correct.
a)(2,17)(15,20)(10,12); b)(-2,3,1)(11,-1,-2)(13,-3,5)

In the following problems, a and b are scalars, A and B are vectors dot product is represented by dp.

5) Prove: dp(A,B)=dp(B,A) .. commutative
6) Prove: dp(aA,bB)=ab*dp(A,B)
7) Prove: dp(A+B,C)=dp(A,C)+dp(B,C) .. distributive

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Intersection of Segments

An Application of the Proportional Point Formula (PPF)

^^^^^^^^^^^^

This section teaches how to make use of the PPF (Proportional Point Formula) to determine where segments intersect. Methods exist to find where lines intersect. The method illustrated here determines where segments intersect. It is possible to find where lines or segments intersect using either method. Sometimes it makes more sense to use a line intersection technique, sometimes it makes more sense to use a segment intersection technique depending on the problem. The method taught here is a great advantage when two segments are given, each defined by two points, especially when the coordinates of these two points are given as variables, not numbers.

Below an example problem and its solution are given to illustrate the segment intersect technique. As always, try to do as much of this problem as you can on your own. If you can't get started or get stuck, look at the solution, just enough to get the needed hint.

Note: Unlike lines, not all non parallel segments in a plane intersect each other. This is because segments have finite length. This method that determines where segments intersect may find they intersect at a point where the segments don't exist, but where they would intersect if their lengths were longer.

I) Determine where the segment (-2,1)(3,4) intersects the segment (3,-1)(1,6).

The Solution

We let V1=(-2,1), V2=(3,4), V3=(3,-1), V4=(1,6)

We make use of the proportional point formula to determine where these segments intersect.

V1+?(V2-V1)=V3+??(V4-V3)

Solving for ? and then substituting ? into V1+?(V2-V1) gives the point on segment (-2,1)(3,4) that is also on the segment (3,-1)(1,6), which is the intersection point of the two segments.

Solving for ?? and then substituting ?? into V3+??(V4-V3) gives the point on segment (1,6)(2,1) which is also on segment (-1,0)(3,4) which is the intersection point of the two segments.

Take a time to understand why the previous two paragraphs are true.

```
V1+?(V2-V1)=V3+??(V4-V3)  ->

(-2,1)+?{(3,4)-(-2,1)}=(3,-1)+??{(1,6)-(3,-1)} ->

(-2,1)+?(5,3)=(3,-1)+??(-2,7) ->

-2+?5=3+??(-2)  and  1+?3=-1+??7 ->

5?+2??= 5    3[5?+2??= 5]     15?+6 ??= 15    15?=15-6??
3?-7??=-2 -> 5[3?-7??=-2] ->  15?-35??=-10 -> 15?=-10+35?? ->
15-6??=-10+35?? -> 25=41??  -> ?? = 25/41

from 5?+2??=5, we get 5?=5-2?? -> ?=(5-2??)/5
-> ?={5-2(25/41)}/5 ->

?= 31/41

The intersection point of two segments = V1+?(V2-V1) =

         31                   ( 73   134 )
(-2,1)+ ----{(3,4)-(-2,1)} = (----,-----)
         41                   ( 41   41  )


Checking our answer, the intersection point of the two segments is
also is V3+??(V4-V3) =

         25                    ( 73   134 )
(3,-1)+ ----{(1,6)-(3,-1)} =  (----,-----)
         41                    ( 41   41  )
```

1) Find the intersection point of these two segments, (-1,2) (5,0) and (1,5) (3,-1). Find where the segments intersect using the following two techniques. a) Make use of the PPF; b) A classical method used to determine where lines intersect (see Section 1).

2) Find where the segments (a,b) (c,d) and (e,f) (g,h) intersect. Using the following two techniques. a) Make use of the PPF; b) A classical method used to determine where lines intersect (see Section 1).

After doing the above two problems, you should have an idea of the strengths of the classical method vs the vector method (PPF). In the problem below, use the method of your choice.

3) S is a square. Each of the midpoints of the sides of S is associated with another point on S as follows. From any of these midpoints, go around S clockwise past the first corner to the second corner. This second corner is the point associated with that given midpoint. Each of these midpoint – corner point combinations define lines. There are 4 such lines. These lines enclose a quadrilateral within S. Prove this quadrilateral is a square and find the ratio of its area with respect to the square S.

^^^^^^^^^^^^

Vector Rotation

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The following discussion on vector rotation is not accompanied by followup problems after each new idea is presented. Therefore it is important the student read carefully and make an extra effort to understand the material. It may be a good idea to read the material more than once.

Vector rotation or rotation in general is a subject usually covered in trigonometry not coordinate geometry. We break "this rule" only slightly in that only vector rotation by +90deg or by -90deg is covered here. The development of the method that rotates vectors by 90 degrees is very simple and does not stray from the spirit of coordinate geometry very far. In trigonometry, rotation by any angle is accomplished and the development of this method is rather involved. The reason we bother to develop a method to rotate vectors by 90 degrees, is so we can develop a simpler method to prove Napoleon's theorem using coordinate geometry. (See the Challenging Problems section). Classical coordinate geometry can be used to prove Napoleon's theorem, however our use of vectors makes this proof much easier. The use of trigonometry allows a more concise and elegant proof of Napoleon's theorem than will be possible with the coordinate geometry method we will use in this book. A trigonometry proof of Napoleon's theorem is presented in the trigonometry answer book.

Definition - Sign of Angle Rotation
A positive rotation or rotation by a positive angle is a rotation that is counter clockwise. A negative rotation or rotation by a negative angle is a rotation that is clockwise.

If a vector is rotated, its magnitude remains the same, but its direction changes. For example the vector <r,o> rotated by 90 degrees, becomes <r,o+90>, the vector <1,0> rotated by 90 degrees becomes <0,1>.

Where o1,o2,o3 are angles, (o1),(o2),(o3) are vector operators, which act upon vectors to rotate them by the angles o1,o2,o3

respectively. For example if V is a vector, (o1)V is the vector V rotated by the angle o1. There are expressions such as (o2)(o1)V where the (o2) seemingly has no vector to act upon. (o2)(o1)V is interpreted as follows. o1 acts on V creating the vector (o1)V, o2 then acts the vector (o1)V to create the vector
(o2){(o1)V} = (o2)(o1)V = (o2+o1)V.

(o1+o2) is a vector operator which rotates vectors by the angle o1+o2. (o1+o2+o3+...+oi) is a vector operator which rotates vectors by the angle o1+o2+o3+...+oi.

The following are vector properties, dealing with vector rotation, that are accepted here as postulates. It is not difficult to visualize the truthfulness of any of these postulates. The reason these postulates are presented here is to provide a basis for the proof that rotation of the vector <a,b> by +90 degrees is the vector <-b,a>, and rotation of the vector <a,b> by -90 degrees is the vector <b,-a>. The student should take the time to become familiar with these postulates and to understand each of them at an intuitive level.

Vector Rotation Properties (2 dimensional)

1) If V is a vector then (o1)V is a vector . . .	closure rotation
2) (o1)[the null vector]=[the null vector] . . .	null vector rotation
3) If o1 = 0,(o1)V=V	zero rotation
4) (o2)(o1)V=(o2){(o1)V}=(o2+o1)V	compounded rotation
5) (o3)(o2)(o1)V=[(o3)(o2)](o1)V=(o3)[(o2)(o1)]V	associative rotation
6) (o2)(o1)V=(o1)(o2)V	commutative rotation
7) if (o1)V1=(o1)V2 then V1=V2	reversibility of rotation
8) (-o1)(o1)V=V	negative rotation definition
9) V1,V2 and V3 are vectors. If V1=V2 and V2=V3, then V1=V3	transitive property of equality

Note: Rotation is commutative in 2 dimensions, not 3 dimensions.

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Vector Rotation by +90 degrees and by -90 degrees (2 dimensional)
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Definition - Vector Operator "&"
The vector operator "&", is defined with respect to a vector in Cartesian form as &<a,b>=<-b,a>.

Theorem
The vector operator “&” rotates vectors by +90 degrees.

Proof

The distance formula tells us that the magnitudes of <a,b> and &<a,b>=<-b,a> are the same, i.e. sqrt{(a)^2+(b)^2}=sqrt{(-b)^2+(a)^2}. Which means the "&" operator leaves the magnitude of any vector unchanged.

We now assume a and b are both arbitrary positive numbers. This implies that <a,b> is an arbitrary vector not aligned with either the x or y axis. The vector <a,b> contains the points (0,0) and (a,b). The equation of the line containing these points is y=(b/a)x. The vector <-b,a> contains the points (0,0) and (-b,a). The equation of the line containing these points is y=(a/{-b})x. The slopes of these two lines multiplied by each other equals -1, therefore these two lines are perpendicular. Therefore the vectors <a,b> and <-b,a> are perpendicular. Therefore the "&" operator, rotates the vector <a,b> by +90deg or -90deg. That a and b are positive numbers means that <a,b> is in the 1st quadrant.

```
    <a,b> .  .  .  .  .  .  is in the 1st quadrant.
   &<a,b>= <-b,a>  .  .  .  is in the 2nd quadrant.
  &&<a,b>=&<-b,a> =<-a,-b>  is in the 3rd quadrant.
 &&&<a,b>=&<-a,-b>=<b,-a>   is in the 4th quadrant.
&&&&<a,b>=&<b,-a> =<a,b>    is in the 1st quadrant.
```

Given that a and b are arbitrary positive numbers, the previous five lines prove that the "&" operator rotates any vector lying in any of the four quadrants by a positive angle, not a negative angle. Having already established that the "&" operator rotates any vector by +90deg or by -90deg, it is now proved that the "&" operator rotates any vector in any of the four quadrants by +90 degrees,
(not -90 degrees).

Next we establish that the "&" operator rotates any vector aligned with any of the axis by +90 degrees.

```
    <a,0>  .  .  .  .  .  . is aligned with the positive x axis
   &<a,0>= <0,a> .  .  .  . is aligned with the positive y axis
  &&<a,0>=&<0,a> =<-a,0>  . is aligned with the negative x axis
 &&&<a,0>=&<-a,0>=<0,-a>  . is aligned with the negative y axis
&&&&<a,0>=&<0,-a>=<a,0>   . is aligned with the positive x axis
```

Given that a and b are arbitrary positive numbers, the previous 5 lines prove that the "&" operator rotates any vector that is aligned with any axis by +90 degrees.

Given that the "&" operator rotates all vectors in any quadrant by +90 degrees and all vectors aligned with any axis by +90 degrees, it is now proven that the "&" operator rotates ALL vectors by +90 degrees.

Proof Complete

How about vector rotation by -90 degrees? We label the operator that rotates all vectors by -90 degrees as "-&". The task now is to discover how this operator acts on a general vector in Cartesian form. Graphically it is apparent that if any vector is rotated by +90 degrees, then multiplied by -1, is the same as rotating that vector by -90 degrees. (The student should take the time to demonstrate to themselves this is true). Therefore we propose that -&<a,b>=-1&<a,b>=-1<-b,a>=<b,-a> or -&<a,b>=<b,-a>. Coincidentally -&<a,b> is the vector &<a,b> multiplied by -1. Now we set out to prove formally what we already "suspect", i.e. that -&<a,b>=<b,-a>.

Proof that -&<a,b>=<b,-a>

&(-&)<a,b>=<a,b> - definition of negative rotation
&<b,-a>=<a,b> - definition of the & operator
&(-&)<a,b>=&<b,-a> - applying transitive property of equality to the previous two lines of this proof
-&<a,b>=<b,-a> - applying "reversibility of rotation" to the previous line of this proof

proof complete

Therefore, by definition -&<a,b> is the vector <a,b> rotated by -90 degrees and in the previous theorem we proved that -&<a,b>=<b,-a>

In Summary: &<a,b>=<-b,a> "&" rotates any vector by +90 degrees
-&<a,b>=<b,-a> "-&" rotates any vector by -90 degrees
&<r,o>=<r,o+90 deg>
-&<r,o>=<r,o-90 deg>

Vector Rotation Problems

1) Calculate <5,-7> rotated by the following angles. a) +90 degrees; b) -90 degrees; c) 270 degrees, d) -90 degrees; c) 180 degrees; d) -180 degrees;

2) Simplify the following.
a) &<m,n>; b) -&<h,-k>; c) -&[&<s,t>]; d) &[-&<m,k>]; e) &&<r,o>.

3) Verify the following by performing the calculations indicated.
a) (-1)*&<r,t>=-&<r,t>; b) &&&<a,b>=-&<k,v>; -&<a,b>=(-1)*&<a,b>

4) Calculate the following.
a) <3,4>+&<3,4>; b) <-1,7> + 1/3*(-&)<-1,7>; c) <a,b>+&&<a,b>

5) The vector <1,5> plus 2 times this vector after it is rotated by +90 degrees. What are these coordinates?

6*) Start at the point P1(-1,-2), go 2/3 way towards the point P2(3,8). Make a turn of +90 deg (turn left) and go a distance equal to 2/5 of the distance from P1 to P2. What are your coordinates?

7*) Find a possible set of coordinates for A,B and C an equilateral triangle. Start at A, from there go half way towards B. From there turn towards C (this is a (+/-)90 degree turn) and go sqrt(3)/6 * the distance from A to B. This point is the centroid of this triangle.

8*) Start at the point P1(-1,-2), go 2/7 way towards the point P2(3,8). Make a turn of -90 deg (turn left) and go a distance of 3. What are your coordinates?

9) Where A and B are vectors, prove that &(A+B)=&(A)+&(B).

10) Where &<a,b>=<-b,a>, prove that the "&" operator rotates vectors by +90 degrees.

11) Where -&<a,b>=<b,-a>, prove that the "-&" operator rotates vectors by -90 degrees.

Note: Important vector subjects not taught in Appendix 2 are
1) Cross Product .. This is taught in the Honors Trigonometry book.
2) Rotation of vectors by an arbitrary angle. This will be taught in the Honors Coordinate Geometry (Analytic Geometry section).

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Derivatives, (No Calculus)
Min Max Problems, Dot Product

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As we are aware, non vertical lines have slopes. The line y=mx has a slope of m. 'Curved' functions also have slope, different points of a curved function have different slopes. For example (speaking intuitively, because slope of a curved function at a point has not been defined yet), the slope of the parabola y=x^2 has a slope of 0 at the point (0,0) and has other slopes at other points of the parabola. Calculus gives a formal definition of slope of a curved function which we will not repeat here.

Calculus is the branch of mathematics one would normally turn to find the slope of curved functions at a point or to find the slope of a curved function at every point of the function. From Calculus for example we know the slope of the parabola y=x^2 has a slope of 2x at the general point (x,x^2). Therefore (for example) the slope of the parabola y=x^2, at the point (5,25) is 10. We will be proving all of this and more using coordinate geometry techniques.

By making use of Calculus techniques, it is possible to find the slopes of virtually every curved function that can be expressed in an equation. Making use of coordinate geometry techniques it is possible to find the slope (at a point) of any conic section. Conic sections are parabolas, circles, ellipses and hyperbolas. We have not yet studied ellipses or hyperbolas, so we will not be finding the slope of these functions in this section. [A future version of this book will include a section of conic sections].

In this section we will be finding the slope (at a point) of circles and parabolas (whose axis are vertical) using coordinate geometry techniques, by making use of the following definition.

Definition: Slope of Curved Graph: If a line with a slope of m intersects any one of certain graphs at a point, and at only that point, then the slope of the graph at that point is m.

(This definition applies to any circle, and any parabola whose axis is vertical).

Take a few moments to ensure yourself that this definition of slope makes (intuitive) sense. (Draw a picture). It is not (yet) proven that this definition of slope is consistent with the universally accepted definition of slope of a curved function, which is given in Calculus. We will not prove this here, because proving this would require knowledge of Calculus and more specifically the limit, and in this book, we do not study Calculus.

Example Problem (Try solving this before looking at the solution).

What is the slope of the parabola $y=x^2$ at the point (5,25)?

Solution

The general line going through the point (5,25) is $y-25=m(x-5)$ or $y=mx+25-5m$.

we have

$y=x^2$ .. parabola
$y=mx+25-5m$ .. line

This line and this parabola intersect where

$x^2 = mx+25-5m$ .. or where
$x^2-mx+(5m-25)=0$

Solving for x by applying the quadratic formula we get the x coordinate(s) where the line and the parabola intersect, which is ..

$$x = \frac{m \pm \sqrt{m^2-4(5m-25)}}{2}$$

The slope of the parabola at the point (5,25) is the slope of the (non vertical) line that passes through this point and intersects the parabola at only one point. We see this line and this parabola intersect at one point only if $\sqrt{m^2-4(5m-25)}=0$. Solving for m..

$\sqrt{m^2-4(5m-25)}=0$ -> .. solving for m

$m^2-4(5m-25) = 0$ ->

$m^2-20m+100=0$

solving for m by applying the quadratic formula we have

$$m = \frac{20 \pm \sqrt{20^2-4*100}}{2} \;->$$

$m = 10$

Therefore the slope of the parabola $y=x^2$ at the point (5,25) is 10.

Finding Slopes of Curved Graphs Problem Set

1) a) What is the slope of the parabola y=x^2 at the point (-2,4)? b) Graph this parabola and the (non vertical) line through the point (-2,4) that intersects the parabola only once.

2) a) What is the slope of the parabola y=x^2 at the point general point (x,x^2)? b) Make use of the result you calculated in part 'a' to determine the slope of the parabola y=x^2, at the following two points, (-2,4) and (3,9).

3) a) What is the slope of the parabola y=x^2-6x+11 at the point (2,3)? b) Graph the parabola y=x^2-6x+11 along with the (non vertical) line that intersects the parabola at (2,3) only once.

4) What is the slope the parabola y=ax^2+bx+c at the general point (x,ax^2+bx+c)?

5) a) Determine the slope of the parabola y=x^2+x-6 at the point (5,24) using the following method. Show that the parabola y=x^2+x-6=0, is the parabola y=x^2 shifted by an amount in the x direction and in the y direction. Now determine the slope of the function y=x^2 at an appropriate point, such that this slope will be the slope we seek. b) Check your answer by doing part 'a', using the 'regular' method taught in this book.

6) a) Make use of the method taught in this section to determine the slope of the circle x^2+y^2=5 at the point {4,3}. b) From classical geometry we know that a line tangent to a circle at a point and the radius of this circle touching this point of tangency are perpendicular. Make use of this fact to determine the slope of the circle x^2+y^2=5 at the point {4,3}. Does this answer agree with the answer you got in part 'a'?

7) a) Using the technique taught in this section, prove that the slope of the circle x^2+y^2=r^2 at the point (x,y) is -x/y. b) Make use of classical geometry to do this problem again.

The Derivative

If the slope of the function f(x) at the general point {x,f(x)} is f'(x), then f'(x) is the derivative of the function f(x).

In this section (and also in Calculus), where f(x) is a function, the derivative of f(x) is denoted as f'(x). Where f(x) is a function of x, f'(x) is also a function of x. It is left to the student to prove this if they wish. [In our case this can only be done for the upper half of circles*, the lower half of circles* and parabolas whose axis is vertical]. The functional value of f'(x) at x, is the slope of the function f(x), at the general point {x,f(x)}.

*Circles are not functions, (When using Cartesian coordinates) therefore circles do not have derivatives. However the upper half of a circle is a function, so is the lower half of a circle. Therefore the upper half (and the lower half) of a circle has a derivative.

<u>Derivative Problem Set</u>

1) What is the derivative of the parabola a) y=x^2; b) y=2x^2-12x+23; c) Make use of your answer in part 'b' to determine the slope of the parabola y=2x^2-12x+23 at the point (1,13).

2) a) Determine the derivative of the parabola y=Ax^2+Bx+C; b) Make use of the derivative of y=Ax^2+Bx+C that you calculated in part 'a' to determine the derivative of y=2x^2-12x+23 and y=x^2. Compare these answers with your answers in problem 1.

3) Determine the derivative of the parabola y=(x-a)^2+b

4) For what value(s) of x does the derivative of the parabola y=2x^2-12x+23 have a value of a) 8?; b) 0? {see problem 1) b)}

5) A parabola P can slide up and down the y axis at will. When the vertex of this parabola is at the origin, its equation is y=x^2. If it is presently very high up on the y axis and is moving down, what will its equation be when it comes to rest by touching, (becoming tangent to) the pair of lines, y=5x and y=-5x?
Hint: Do problem 4 first.

6) a) Determine the derivative of the upper half of the circle x^2+y^2=r^2; b) Determine the derivative of the lower half of the circle x^2+y^2=r^2.

7) Determine the derivative of the upper half of the circle (x-a)^2+(y-b)^2=r^2.

8) a) Make use of the techniques taught in this section (you may make use of the results of the previous two problems if you wish) to find the two points where the circle centered at the origin with radius of 1, has a slope of 3. b) Make use of classical geometry to do this problem again, verifying that the answer you got in part 'a' is correct.

9) A circle of radius 1 is centered at a point on the positive y axis, and is tangent to the lines y=4x and y=-4x. What is the equation of this circle?

Relative Maximums and Relative Minimums

If a point 'p' of a function 'f' lies in the interior of a horizontal segment (meaning 'p' is a point of the segment but 'p' is not an end point of the segment), and if the y coordinate of this segment is greater than or equal to all y coordinates of the points of 'f' that have the same domain as this segment, then the y coordinate of point 'p' is a relative maximum of the function 'f'.

The definition of relative minimum is similar to the definition of relative maximum. It is left to the student make use of the definition relative maximum to construct a valid definition of relative minimum.

If a maximum of a function f occurs at a point p={x,f(x)}, then the y coordinate of p is greater than or equal to the y coordinates of all other points of the function. If a relative maximum of a function occurs at a point p={x,f(x)}, then the y coordinate of p is greater than or equal to the y coordinates of all points of f(x) that are near the point p. If p is in the interior of some horizontal segment, then all points of the function f(x) that have the same domain as the horizontal segment are deemed as being near to the point p.

Similarly, if a minimum of a function f occurs at a point p={x,f(x)}, then the y coordinate of p is less than or equal to the y coordinates of all other points of the function. If a relative minimum of a function occurs at a point p={x,f(x)}, then the y coordinate of p is less than or equal to the y coordinates of all points of f(x) that are near the point p.

Take the time to digest the definitions and the meanings of maximum and relative maximum and minimum and relative minimum of a function.

Where y=f(x) is a parabola (whose axis is vertical), or the upper or lower half of a circle. If f'(x)=0, at x=a. Then a relative maximum or relative minimum of f(x) exists at x=a.

We accept the preceding paragraph as a postulate, although in Calculus this is proven, for a wide class of functions. Take the time to understand the truthfulness of this postulate intuitively.

If f'(x)=0, at x=a, then x=a is the location of a relative maximum or a relative minimum of y=f(x). By graphing the function y=f(x) about x=a, you can determine if a relative maximum or relative minimum exists at x=a.

Take the time and effort to verify that for functions that are parabolas (whose axis are vertical) or upper half of circles or lower half of circles, .. relative maximums are maximums. Also

relative minimums are minimums. This is not true for all functions.
Relative Minimum Relative Maximum Problem Set

1) I) Make use of the derivative to determine locations of relative maximums and relative minimums of the following functions.
a) y=-x^2; b) y=2x^2-12x+23; c) -x^2+6x-11; II) Make use of the shifting theorem to do these problems a different way, thereby checking your answers.

2) I) Make use of the derivative to determine locations of relative maximums and relative minimums of the following functions.
a) the upper half of the circle x^2+y^2=1;
b) the lower half of the circle x^2-2x+y^2-4y+1=0;
c) the upper half of the circle (x-3)^2+(y+4)^2=25
II) Make use of shifting theorem do these problems a different way, thereby checking your answers.

Definition Dot Product 2d: Where (x1:y1) and (x2:y2) are slopes (in ratio form), the dot product of these slopes is x1*x2 + y1*y2

Definition Dot Product 3d: Where (x1:y1:z1) and (x2:y2:z2) are slopes (in ratio form), the dot product of these slopes is x1*x2 + y1*y2 + z1*z2

3) Dot Product : (2 dimensions)
a) Given the line L=(x1,y2)+t(a,b), and the point p(x2,y2). Find the point k on L that is closest to the point p. (You will be making use of the derivative and the concept of relative minimum to find the minimum distance between the point and the line).
b) Prove that the dot product of the slopes of the lines pk and L equals 0. c) Make use of the results in 'a' and 'b' to prove that in two dimensions, if the dot product of two slopes is 0, then the slopes are perpendicular to each other. d) Prove that if slopes in two dimensions are perpendicular, their dot product is equal to zero.

4) Dot Product : (3 dimensions)
b) Given the line L=(x1,y2,y3)+t(a,b), and the point p(x2,y2,y3). Find the point k on L that is closest to the point P. (You will be making use of the derivative and the concept of relative minimum to find the minimum distance between the point and the line).
b) Prove that the dot product of the slopes of the lines pk and L equals 0. c) Make use of the results in 'a' and 'b' to prove that in three dimensions, if the dot product of two slopes is 0, then the slopes are perpendicular to each other. d) Prove that if slopes in two dimensions are perpendicular, their dot product is equal to zero.

5) Find a line perpendicular to line t(1,2,3).

6) Find one line perpendicular to lines t(1,2,3) and t(3,-4,7).

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12 Challenging Problems

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There are some proofs in this section that use vector techniques. Studying vectors (appendix 2) would help to do these problems.

0) Keeping in mind the following problem previously done, {A man stands 24m from the base of a flag pole, the angle from the man's feet to the top of the flag pole is 1.23 deg, how high is the top of the top of the flag pole? and how far are the man's feet from the top of the flag pole}? In this problem it is possible to measure the distance from the observer to the base of the flag pole. Devise a practical method, (using ability to measure both distance and angles) to determine the height of a mountain, take note it is not possible to measure the distance from the observer to the "base" of the mountain peak, nor are laser range finders allowed, i.e. it is not possible to measure the distance from the observer to the mountain peak directly. Answer is below in next problems, try doing this yourself before looking.

1) A tourist measures the slope of elevation to a mountain top to be 0.306. The tourist then drives 3 miles on level ground towards the mountain and measures the angle of slope to the mountain top to be 0.601. How high is the mountain? How far was the observer from the mountain top when taking each of these angle measurements?

2) The slope from a group of tourist to a mountain top is m1. The tourists travel on level ground directly towards the mountain a distance of d, at which time the slope from the tourists to the mountain top is m2. Prove the height of the mountain is h=d/(1/m1-1/m2).

3) The slope from a group of tourists to a mountain top is m1. The tourists travel towards a distance d in a straight line and on level ground as follows. Directly towards the mountain, through a tunnel underneath the mountain and then directly away from the mountain. At which time the slope from the tourists to the top of the mountain is m2. Prove the height of the mountain is h=d/(1/m1+1/m2).

4) Point P lies on the line y=x. The coordinates of Q and T are (-1,8) and (8,1) respectively. The sum of the distances QP and TP is 18 units. Determine the coordinates of P. (after doing this analytically, have student do using the -find the zero’s feature- on a calculator).

5) In a rectangular room, 3 meters by 5 meters, what is the longest 1 meter wide rectangular carpet that can be placed on the floor?

6) The sides of a triangle are 7, 8 and 9, find one possible set of coordinates for vertices of this triangle.

7) a) A triangle has sides 4,5 and 6, find its area. b) A triangle has sides A,B and C, find its area.

8) The midpoints of the sides of a triangle have coordinates (3,1), (-1,2) and (1,-3). Determine the coordinates of the vertices of the triangle.

9) A line through the point A(1,2) intersects the line 2y=3x-5 at P. This line intersects the line x+y=12 at Q. if AQ = 2AP, find the coordinates of P and Q. (AQ and AP are segment lengths).

10) An astronaut tall enough to have eyes 6 feet above the ground is exploring a spherical asteroid 5 miles in circumference and gets lost from his space ship. The spaceship's beacon is 21 feet above the ground. When the astronaut is first able to see the spaceship's beacon over the horizon of the asteroid, what is the linear distance between his eyes and the beacon of the spaceship?

11) Determine (Approximately) Size of the Earth
A sensor is able to determine the time of sunset. (defined here as the time when the upper edge of the sun disappears beneath the horizon, from the vantage point of the observer). When the sensor is at ground level it detects that a sunset has occurred, and it is quickly raised to 5 feet above ground level. The sensor again detects the occurrence of a sunset 9.564 seconds later than before. Assuming the sensor is at a place and time where the sun passes directly overhead at noontime, calculate the circumference of the earth.

Using only the mathematical techniques of coordinate geometry, it is not possible to solve this problem with exactness. However this problem can be solved with a high degree of precision if one makes the following simplifying assumption, ... the distance of shortest possible path between two points on the earth's surface, when moving along the earth's surface equals the straight line distance between the two points ... This assumption is never exact, but is good enough when the two points are sufficiently close to each other. The use of simplifying assumptions is a powerful problem solving tool frequently used in science and engineering. Simplifying assumptions usually always result in an answer having error and are useful only when the error introduced is sufficiently small. The error introduced by this simplifying assumption in this problem is less than 1/2 foot. Given that the circumference of the earth is about 25,000 miles this is indeed a very small error. If trigonometry (which is the math subject that comes after coordinate geometry) is used to solve this problem, this simplifying assumption is not necessary and the less than 1/2 foot error is not introduced.

12) The points B and C of triangle ABC lie on the lines 3y=4x and y=0 respectively. The line BC passes through (2/3,2/3). If O is the origin and AOBC forms a rhombus, find the co-ordinates of A.

13) Prove: The sum of, the square of each of the distances, from the vertices of a square to any line passing through its center, is a constant.

14) Prove Angles Are Equal
The following rays have a common end point and exist in the 1st quadrant. Ray A has a slope of 3, ray B has a slope of 1, ray C has a slope of 2 and ray D has a slope of 3/4. Prove that the angles formed by the ray A and ray B equals the angle formed by ray C and ray D.

15) a) Given the line y=-7 and the points (2,2) and (-1,4) find the circle that passes through both points and is tangent to the line. b) Given the line y=x+6 and the points (2,2) and (-1,4), find the circle that passes through both points and is tangent to the line.

16) Two circles A and B (not necessarily of the same size) are externally tangent to each other and to a line at the point T. P is a point on this line and through P there are two lines, one is tangent to circle A at point Ta and the other is tangent to circle B at point Tb. Prove that the lengths of the segments P Ta and P Tb are equal.

17) a) Find the point(s) on the line y=3x-10 that are a distance 5 from the parabola y=x^2. b) Find the point(s) on the parabola y=x^2 that are a distance 6 from the line y=3x-10. (See the Derivatives section).

18) Locus Problem - Ladder Problem
A ladder is in a upright vertical position adjacent to a wall. The bottom of the ladder is pulled directly away from the wall until the ladder is in a horizontal position, while the ladder is being pulled away from the wall the top of the ladder maintains contact with the wall. Prove that while this is happening, the middle of the ladder traces out a quarter circle.

19) A circle centered at the origin has radius r1. A circle centered at (0,h) has radius r2, h>(r1+r1) a) How many lines are tangent to both circles? Answer question "a" before reading problem "b", STOP. b) Find the equations of the four lines that are tangent to both circles.

20) A circle of radius 1 rolls down the line y=(1/3)x towards the origin until it also makes contact with the line y=(1/2)x, at which time it comes to rest. Once the circle is at rest, what is its equation?

21) One of the rays forming both angles A and B has its end point at the origin and lies along the positive x axis. The other rays forming each of these angles lies in the first quadrant. Angle A has a slope of 1/2. Angle B has a slope of 1/3. a) Determine the slope of angle (A+B). b) Angle A has a slope of m. Angle B has a slope of n. Prove that angle (A+B) has a slope of (m+n)/(1-m*n). Hint: For this problem, assume A>0deg, B>0deg, A+B<90deg. (These restrictions on A and B are not mathematically necessary, but were given to make the problem easier to solve. In trigonometry, a different (better) proof will be used to prove this angle addition formula for the general case. (No restrictions on angle sizes).

22) Radius of Largest Circle Inside Triangle
a) A triangle has sides of length 4,5 and 6. Calculate the radius of the largest circle that will fit inside this triangle. b) Prove that the radius of the largest circle that will fit inside a triangle equals 2*(Area of Triangle)/(Perimeter of Triangle).

11.23) Radius of Circle Inscribing Triangle
A triangle has sides, s1, s2 and s3. Prove that the radius of the circle that passes through all three vertices of this triangle is r={s1*s2*s3}/{4*Area of the Triangle}.

11.24) L1 and L2 are parallel lines. L1 is above L2. The distance between them is 1. L1 intersects the y axis at 7 and L2 intersects the x axis at 3. a) What is the slope of L1 and L2?; b) What is the x intercept of L1?; c) What is the y intercept of L2?

11.25) The perimeter of a right triangle is 60 inches and the altitude perpendicular to the hypotenuse is 12 inches. What are the lengths of the sides of the triangle?

26*) The following set of points are vertices of rhombus given in counter clockwise order, starting with the lower left vertex, then lower right, then upper right, then upper left. a) Make use of the distance formula to determine all vertices. b) Using vector addition, do this problem again. (0,0)(x2,0.9740)(3.4266,y3)(x4,2.3933) Note: This problem is much easier to do when using vector addition.

27) a) A circle contains the point (4,-1) and is tangent to the line y=2x-6. Give a general equation for the family of all such circles. b) What member(s) of this family have a radius of 12?

11.28) Ladder Problem

Two buildings, A and B stand next to each other forming an alleyway. There are two ladders in the alley. The bottom of one latter sits against the base of building A, and leans over onto building B. The bottom of the other ladder sits against the base of building B, and leans over onto building A. One ladder is 3 meters long the other ladder is 4 meters long. These two ladders cross each other, (they touch at the point where they cross) at a point 1 meter above the ground. How far apart are the buildings?

29*) Assume that the shortest path between any two parabolas (that don't intersect) is along a segment perpendicular to both parabolas. Make use of this fact to find the distance between the following two parabolas. y=2(x-1)^2-2 and y=-(x+3)^2+5.

30*) The Parabola as an Unexpected Vice.

Line L has a normal or home position of x=0. It is constrained to lie on the origin (0,0), but is free to rotate about the origin assuming nothing is preventing it from doing so. Assuming line L is not free to penetrate (intersect) the parabola y=x^2 (except at the origin) because both the parabola and the line are made of hard non permeable materials, <u>prove that the presence of the parabola y=x^2 prevents line L from rotating from its home position</u>. This is surprising since there is seemingly plenty of room (rotation wise) between the parabola and the line, and the higher one goes the distance between the line and the parabola increases. <u>This is a very interesting geometry theorem</u>.

11.31) Q: a) If the graph of y=f(x) is stretched like an accordion along the x axis by a factor of 2, has it been compressed or stretched? b) If the graph y=f(x) is stretched like an accordion along the x axis by a factor of 1/2, has it been compressed or stretched? c) If I get 2 times closer to you than before, am I now closer to you or farther away? d) If I get 1/2 times closer to you than before, am I now closer to you or farther away? e) If I get 1 times closer to you than before, am I closer to you than before? Am I farther to you than before?

The Compression Theorem

If any function y=f(x) is compressed horizontally, towards the y axis by a factor of j, such that every point of the function becomes j times closer to the y axis, ..AND.. if this function is compressed vertically, towards the x axis by a factor of k, such that every point of the function becomes k times closer to the x axis, then this function becomes .. y*k = f(j*x) or y=f(j*x)/k.

f) Take the time to understand this theorem.
g) y=f(x) compressed by 2 about the x axis becomes what function?
h) y=3x^2 stretched by 3 about the y axis becomes what function?
i) Using the Shifting theorem as a guide (proven previously in

this book), prove the Compression Theorem for yourself.

11.37) All Parabolas are Similar (They are the same shape)
If two geometrical objects have the same shape, but not necessarily the same size, they are said to be similar. Triangles that have the same angles are all similar. All circles are similar. All squares are similar. Not all rectangles are similar, not all triangles are similar. Prove that all parabolas are similar, i.e. prove that y=ax^2 is similar to y=bx^2. (Hint: To do this make use of the "Compression theorem" given in the previous problem). Note: Parabolas may be defined as the set of points that are equidistant from a line, and a point not on that line. All lines together with a point not on the line are similar regardless of the distance between the line and the point. (Can you see why)? Given this, it seems reasonable that all parabolas should be similar. This is not a proof that all parabolas are similar, but is another way to see why they ought to be similar.

-) Locus Problem: Prove that the set of points p, such that distance from p to (-2,0) + distance from p to (2,0) = 6, is circle that is compressed towards the x axis by some factor k where (k>1). (see compression theorem). This compressed circle is called an ellipse.

32*) Prove: If two chords of a circle intersect and are of equal length, and if their point of intersection divides one of these chords into segments of lengths A and B, then this point of intersection divides the other chord into segments of lengths A and B also.

11.33) Locus Problem: Prove that the set of points that is c (c!=0) times as close to one line as to the other line is a set of two lines.

34) Locus Problem - Two perpendicular lines intersect at a point P and pass through the ends of a segment. Prove that the set of all such points (P) is a circle. [This segment is the diameter of the circle].

35) Locus Problem - Two perpendicular lines intersect at a point P and are tangent to a parabola. Prove that the set of all such points (P) is a line. [This line is the directrix of the parabola, see problem 47 to learn more about the directrix of a parabola].

36) Locus Problem - Prove: The union of the midpoints of -all chords of a parabola that pass through a fixed point-, is a parabola.

38) Right Triangle Inscribed in Parabola
A right triangle RST with hypotenuse ST is inscribed in the parabola y=x^2 so that R coincides with the vertex of the parabola. If ST intersects the axis of the parabola at Q, then show that Q is independent of the choice of right triangle. i.e. (show that the hypotenuse of all such right triangles must pass

through Q).

39*) Power Theorems

(a) c1 is a chord of a circle having end points A and B and c2 is a chord of the same circle having end points C and D. c1 and c2 intersect at point M, prove the chords AM * MB = CM * MD. (b) P is a point outside a circle. Two rays, r1 and r2 having P as their end point pass through the circle. r1 intersects the circle at points A and B st. P-A-B, r2 intersects the circle at points C and D st. P-C-D. Prove PA*PB = PC*PD. (c) P is a point outside a circle. Two rays, r1 and r2 have P as their end point, r1 intersects the circle at points A and B st. P-A-B, r2 is tangent to the circle at point C. Prove the chords PA*PB = (PC)^2. d) If you proved a,b and c using separate proofs, now prove a,b and c using only 1 proof.

40*) Power Theorems (Converses)

(a) AB and CD are segments that intersect at E. Assuming AE*EB = CE*ED prove that points A,B,C and D are cyclic. (cyclic means these points are all on a single circle). (b) P is the endpoint of two rays r1 and r2. A and B are points on r1 st. P-A-B, C and D are on R2 st. P-C-D. Assuming that PA*PB = PC*PD prove that points A,B,C and D are cyclic.

Definition: The centroid of a triangle is that point where the 3 medians of the triangle meet.

11.41) Prove: All three medians of any triangle meet at a common point, and this point (the centroid of the triangle) is two thirds of the distance from any triangle vertex towards the midpoint of the opposite side.

11.42) Napoleon's Theorem 1

If equilateral triangles are placed on all three sides of an arbitrary triangle, such that the arbitrary triangle and each of the equilateral triangles share a common side, and the equilateral triangles are pointed outward with respect to the arbitrary triangle, prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the arbitrary triangle's, Napoleon triangle 1.

11.43) Napoleon's Theorem 2

If equilateral triangles are placed on all three sides of an arbitrary triangle, such that the arbitrary triangle and each of the equilateral triangles share a common side, and the equilateral triangles are pointed inward with respect to the arbitrary triangle, prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the arbitrary triangle's, Napoleon triangle 2.

44) Napoleon’s Theorem - Area Difference
Given an arbitrary triangle, prove that the difference of the areas of its Napoleon triangle 1 and its Napoleon triangle 2 equals the area of the original arbitrary triangle. (This is a difficult problem to do using coordinate geometry techniques, much easier using trigonometry).

11.45) Beautiful Cable Support Structure .. Locus Problem
Two rings (circles) of radius r lie in horizontal parallel planes separated by a distance of h. One ring is directly on top of the other, such that the line connecting the centers of these rings (the centerline) is perpendicular to the planes containing the rings. Each ring has on it n (n= 3 or 4 or 5 or ...) evenly spaced hooks. Every hook on the bottom ring has directly above it, a hook on the top ring and every hook on the top ring has directly below it, a hook on the bottom ring. The bottom ring is suspended in the air by cables connecting it to the top ring in the following manner. Each hook of the top ring, connects to two hooks on the bottom ring as follows. A top ring hook connects not to the hook directly below it, but to a hook 1 or 2 or 3 ... hooks to the right AND to the left of the hook that is directly to below it. Question, what is the shape the outline of the cables connecting the two rings? When I was young, there was a community theater called the Valley Music Hall, that had such a setup. The lower ring supported a platform that had stage lights on it. Viewed from the side, (looking at the rings edge), the outline of the cables made a beautiful and interesting shape of what looked like a hyperbola or parabola pointing outward from and "rotating" around line connecting the centers of the top and bottom rings. a) Find the shape of the figure (its equation) being "rotated" around the center line. b) Is this rotated shape a parabola (geometrically similar to y=cx^2), a hyperbola (geometrically similar to x^2/a^2-y^2/b^2=1), or something else? (This shape is a hyperbola)

46) Prove: If a line L is perpendicular to two non parallel lines that lie in a plane, then L is perpendicular to every line that lies in that plane. Hint: See Dot Product in the Vectors Section, also see dot product Derivatives section.

-) Ray 1 has a slope of m1. Ray 2 has a slope of m2. The slope m2 is greater than the slope m1. (m2>m1). The end point of both rays is at the origin and both rays are located in the first quadrant. In an alternate rotated, coordinate system, where the end point of both rays are still located at the origin and ray 1 now lies along the positive x axis, prove that the slope of ray 2 is (m2-m1)/(1+m1*m2).

47) Parallel Reflector (Another Parabola Property)

Whenever light or sound impacts a reflective surface, curved or not, the angle of approach with respect to the surface at the point of impact, is equal to the angle of retreat with respect to the surface at the point of impact. A parabola is the set of all points that are equidistant from a point and a line. This point is referred to as the focus of the parabola, this line is referred to as the directrix of the parabola. If light or sound emanates in all directions from the focus of any reflective parabola and then impacts the parabola, prove that after impact, the light/sound moves along a path that is perpendicular to and away from the directrix of the parabola.

This is how a flashlight works, the shape of its reflector is a parabola rotated about its axis. A light is located at the focus of this rotated parabola. Light emanates from the focus, then reflects off of the reflector of the flashlight, and from there travels in a parallel direction perpendicular to the (rotated) directrix of the parabola. (The rotated directrix is a plane).

A related conclusion to the one being proved here is this. If a signal of light or sound approaches a reflective parabola along a path that is perpendicular to the directrix of the parabola and
.. the parabola is between the signal source and the directrix,
.. then after the light or sound impacts the parabola, it will be reflected towards the focus of the parabola.

This is how a satellite dish works. Its shape is a parabola rotated about its axis. The dish is oriented so that the radio waves coming from the satellite travel in a direction perpendicular to the directrix of the (rotated) parabola, i.e. the satellite dish. Upon impacting the dish, the radio waves are reflected to the focus where an electronic receiver is located and can then pickup the transmission from the satellite. Note: The section "Derivatives (No Calculus)" is a prerequisite for this problem.

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Appendix 1
Triangle Centers

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Note: Some problems in this section may or may not be well suited to do using coordinate geometry, (classical geometry or trigonometry may be better). Explore at your own risk. (Some problems may be excessively complex or long when solved using coordinate geometry).

Definition: Concurrent
Lines are concurrent if they meet at one common point.

Definition: Incenter
The incenter of a triangle is the point where the three angle bisectors meet.

Definition: Centroid
The centroid of a triangle is the point where the three medians meet.

Definition: Orthocenter
The orthocenter of a triangle is the point where the lines that contain the three altitudes.

Definition: Circumcenter
The circumcenter of a triangle is the point where the perpendicular bisectors of the three sides meet.

1) Prove: All three angle bisectors of any triangle intersects at a common point (this point is called the incenter).

2) Prove: The three medians any triangle intersects at a common point G (this point is called the centroid).

3) Prove: The lines containing all three altitudes intersect at a common point H. (this point is called the orthocenter).

4) Prove: The perpendicular bisectors of the three sides of any triangle intersect at a common point C. (this point called the circumcenter).

5) a) Prove that for any triangle, the points H,G and C in the previous three problems are collinear. The line that contains these points is called Euler's line. b) prove CG=1/3*CH.

6) Ceva's Theorem (ver 1) For any triangle ABC with points D,E,F that lie on the segments BC,CA,AB respectively, the lines AD,BE and CF are concurrent if and only if the distances AF,BD,CE,FB, DC,EA satisfy AF*BD*CE=FB*DC*EA. This is true for angle bisectors and medians.

7) Ceva's Theorem (ver 2) For any triangle ABC with points D,E,F that lie on the lines BC,CA,AB respectively, then the lines AD,BE and CF are concurrent if and only if the signed distances AF,BD, CE,FB,DC,EA satisfy AF*BD*CE=FB*DC*EA. This is true for angle bisectors and medians and altitudes.

Note: Read the following to see how negative distances occur in regard to Ceva's theorem. Let M and N be any two different vertices of a triangle, and p be the point that is on line AB, Then

- M sees positive direction as being along line MN and towards N
- N sees positive direction as being along line MN and towards M
- M sees negative direction as being along line MN and away from N
- N sees negative direction as being along line MN and away from M

- Therefore

- if M-F-N .. MF is positive and NF is positive
  (F is on segment MN)
- if F-M-N .. MF is negative and NF is positive
  (F is not on segment MN)
- if F-N-M .. NF is negative and MF is positive
  (F is not on segment MN)

8) For a triangle with vertices A,B and C, with circumcenter O, and with orthocenter H, prove OH=OA+OB+OC. Note: The circumcenter of a triangle is the point where the lines that contain the three perpendicular bisectors of the sides of a triangle meet. The orthocenter of a triangle is the point where the lines that contain all 3 altitudes of the triangle meet.

9) Given any triangle MQR with S being the midpoint of MQ and T being the midpoint of RQ. The median RS and the median MT intersect at P. Prove that the area of triangle PMS equals the area of triangle PRT.

Hint: Working out the solution to this problem is a fairly lengthy process. Variable substitution can make the solution less laborious. Whenever the coordinates of a point are more complex than a multiple of a single variable, renaming the coordinates of that point so they are single variables, i.e. (m,n) can make the problem easier. In the end you have to remember to substitute in the correct values for the coordinates of the point.

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The Story and the Hope of This Book

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In my 10th grade Geometry class in 1973 I gained a life long love of and passion for mathematics. One day the teacher told us that he would prove to us that the angles of all triangles add up to 180 deg. I immediately wondered how he could possibly prove this for all triangles since there was an infinite variety of them. He did prove it and I was satisfied the proof was true. I was impressed that such a thing could be done. After this I saw many other interesting proofs. As a class we were given 3 to 4 proofs every night as home work. As a result, we all (those in my Geometry class) became very capable mathematical problem solvers, and I for one was inspired with a love of mathematics and the power and problem solving ability that studying mathematics imparts to people.

By 2004, my oldest son was taking Geometry. I wanted the same experience for him that I had had. Unfortunately I learned that in the years since I taken mathematics in Junior High and High School, math education had become dumbed down. My son was 3 months into his Geometry class and still hadn't done any proofs. In looking at his book I couldn't find any. The only problems I could find dealt with geometric facts, that were memorized and problems that required very simple reasoning. Gone for the most part were proofs and any medium or hard problems. I contacted the teacher and it was explained to me that Geometry education had changed since the days I had gone to school. Proofs had been de-emphasized, but not to worry, there would be a two week section where they would do proofs.

I was very disappointed. The next school year, I told my son not to take math from the school, that I would teach him his geometry, from the book I used when I took Geometry, "Geometry by Moise and Downs". I complained to the School District Math supervisor, who told me she had heard this complaint before from others, she then asked me if I would be willing to serve on the Geometry text book selection committee. I told her I would be more than happy to. I immediately began to search the internet for a good geometry book (that was still in print). For quite some time I couldn't find one. Geometry education, nation wide had become severely dumbed down.

I contacted the publisher of my old Geometry book to see if Davis County Utah could secure the rights for its use. In the process of trying to do this, the publisher told the Davis county math supervisor, what I was trying to do and she had withdrew my invitation to serve on the committee. I was very disappointed and pleaded to be able to serve, but she was firm.

I had given up hope, thinking all I would be able to do would be to ensure that this dumbing down of Geometry education would not affect my own kids. This to me was not enough, if others had only been concerned about their own kids when I was young, I would not have had the opportunity to get the quality education I was fortunate to get. I found that all areas of math education not just Geometry had been watered down, however Geometry was the most severely impacted. Many elementary schools had given up teaching students to add, subtract, multiply or divide .. fractions or decimals, preferring to use a calculator instead. In Algebra, at least in the school my kids were attending, word problems weren't being taught anymore, and something called the foil method, was being taught to teach kids to multiply monomials i.e. (x+5)(x-1), this is a dumb little method that had taken the place of making use distributive law to solve these problems. The problem with the FOIL method is it is simply a memorized procedure that imparts no number sense and it only works for this case, it will not work for something like (a+b+c)(d+e).

I also found in that in many modern trigonometry books, many proofs are being put in the an appendix where the chance of them being looked at is nil. When I took math, seldom did we ever make use of any mathematical procedure or theorem without first proving it. This gave us the ability to see for ourselves that it is true, and helped to teach us how to do proofs on our own. To teach math, without teaching the proofs provides a shallow math education.

I am an Electrical engineer, in my vanpool there is an other engineer who sometimes substitute teaches math. He told me that when he went to substitute teach Calculus at one of our local high schools, he was told by the lead math teacher, to teach the students to use the chain rule, but not to bother teaching the proof. He got the impression, they wanted to teach to the test to make the school look good, and proofs were not going to be on the AP calculus test. He very much disagreed with this approach and taught the students the proof anyway, then went on to teach them how to use the chain rule. He said that he was never invited back to teach there.

When my oldest son was taking physics, I noticed he was using a 3x5 card with the formulas on it, when doing his home work. I wondered why this should be necessary, and asked him why he had to use the card. He said the teacher let them. I asked him if the physics teacher had ever proven any of these formulas to the students or had the students prove any of these formulas for themselves. He told me no. When back to school night came, I asked the physics teacher why he was just handing the students formulas without proving them or having the students prove or derive them for themselves. He looked at me and said he agreed that he should be doing this and told me he used to do this, however students and parents had complained that he was making the class too hard so he backed off. He explained to me that students today are not as prepared mathematically as we were

when we were going to high school.
Let me say, it wasn't necessary for me or others when I took physics in high school to have a 3x5 card to put the formulas onto, this because the teacher either showed us the proofs or had us prove these formulas for ourselves, and once you prove a formula it becomes yours. If you do forget it you can quickly re-derive it, and then you have it again. Proving a formula helps provide an in-depth understanding of physics, merely handing the kids the formulas means they don't learn why the formulas are true, and they don't learn how to come up with formulas for themselves. The way physics is taught now, provides only a shallow superficial understanding. Unfortunately this has become seemingly necessary because too many of our teachers of mathematics and the publishers of math text books aren't doing their jobs properly.

I could give more examples, but won't here. In short, I feel the math education community in the United States has dropped the ball, they have lost their way.

Why did I write this book and others? My writing these books is a result of me tutoring my own kids. I found while tutoring them that I was coming up with some high quality and very interesting math problems on my own, as well as finding great problems from older books and out of print math books on Google books, I also found some very good problems on various websites on the internet. At first, as I was teaching my oldest, I would dispose of my lesson materials after each lesson. I have 6 kids. It occurred to me that I might not be able to again come up with these great lessons next time when I teach my younger kids, so I decided to keep a written record of each lesson. That's how these books got started. After a short time, keeping records of my lessons took on a life of its own. In teaching my oldest trigonometry (my first book), I had practically written a trigonometry book, when I saw what I had almost done, I then decided to actually do it. After that, I taught my oldest analytic geometry and wrote a book on this also. After that I began this book on coordinate geometry. I have also begun a book on classical geometry. (necessary because the Great Geometry book by Moise and Downs is no longer in print and even though it is a very good book, I can see several areas of possible improvement).

Of the books I have written, this book on Coordinate Geometry is the best compared to what is out there already. It wouldn't surprise me if this is the worlds best Coordinate Geometry book for honors students at the junior high or high school level. (There are some very old Coordinate Geometry books suitable university students that are well suited for secondary students).

To help combat this decline in mathematics education, I decided to make all these books available to everyone, and that has been a major motivation in my putting even more effort into all these books than I would have otherwise. I hope I can make a difference and help

to turn back this tide of mathematics education mediocrity.
If the United States is going to continue to be a world leader in vision, science and technology, we have to give our young people a quality math education where they can be given many opportunities to solve challenging problems and be inspired by the beauty and power of serious mathematics. If we don't turn this around, our decline is inevitable and will manifest itself in the form of fewer innovations, more failed and less ambitious engineering projects, we will have less educated people in general, no matter their field of study. Math education is central. We can turn this around, but for this to happen the math education community will need to stop denying that a problem even exists.

If you are a teacher, or a student and find this book useful, please tell me, I would like to know if this book is helping people, and the cause of math education improvement. If you have suggestions to improve this book, please let me know.

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Curtis Blanco
Bountiful UT

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Answers to Problems

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Note: This answers to problems section is still under construction. Many solutions to problems still need to be added. Also many of these solutions are still in their first draft, i.e. it is possible to improve their wording and even their logic in some cases.

1.1) Given the points E(-4,0), G(3,5), and K(8,-2). Show that line EG and GK are perpendicular by showing the product of their slopes is –1. Check this conclusion graphically, by graphing these lines.

line EG = E(-4,0) G(3,5), the slope of this line is

$$m1 = \frac{y2-y1}{x2-x1} = \frac{5-0}{3-(-4)} = \frac{5}{3+4} = \frac{5}{7}$$

line GK = G(3,5) K(8,-2), the slope of this line is

$$m2 = \frac{y2-y1}{x2-x1} = \frac{-2-5}{8-3} = \frac{-7}{5}$$

$$m1*m2 = \frac{5}{7} * \frac{-7}{5} = -1$$

Therefore EG and GK are shown to be perpendicular as requested.

1.4) The vertices of a triangle are (16,0), (9,2), and (0,0). a) What are the slopes of its sides? b) What are the slopes of its altitudes?

To do this problem, (or any problem of more than a little complexity) draw a picture of the situation. Hint: Most such pictures don't need to be to scale or even oriented correctly (use judgement).

The sides and the associated slopes of the triangle are

s1=(0,0)(16,0) m1 = (0-0)/(16-0)=0
s2=(16,0)(9,2) m2 = (2-0)/(9-16)= 2/(-7)
s3=(0,0)(9,2)   m3 = (2-0)/(9-0)= 2/9

Remember from classical geometry that triangles have 3 altitudes. An altitude can be a segment, or the length of that segment. The word altitude is used for either. The (segment) altitudes of a triangle are defined as follows. One endpoint of the segment is a vertex of the triangle, the other endpoint of the segment is on the side (of the triangle) opposite the segment, and the slope of the segment is perpendicular to the segment that the altitude touches.

We define altitude 1 the altitude whose endpoint is on segment 1
We define altitude 2 the altitude whose endpoint is on segment 2
We define altitude 3 the altitude whose endpoint is on segment 3

Since the side of a triangle and the altitude to it are perpendicular the slopes of the altitude and the corresponding side are negative reciprocals of each other.

Therefore slope of altitude 1 = -1/(m1) = undefined
Therefore slope of altitude 2 = 1/(m2) = 7/2
Therefore slope of altitude 3 = 1/(m3) = -2/9

1.7) Do "a" and "b" graphically before doing them analytically. What value(s) of k will make the lines (k,3)(-2,1) and (5,k)(1,0) a) parallel? b) perpendicular? c) Check answers by graphing lines.

These lines are parallel if their slope are equal and these lines are perpendicular if the product of their slopes equals -1.

a)

Lines are parallel implies ->

slope of (k,3)(-2,1) = slope of (5,k)(1,0) ->

$$\frac{3-1}{k-(-2)} = \frac{k-0}{5-1} \quad ->$$

$$\frac{2}{k+2} = \frac{k}{4} \quad -> \quad \text{(cross multiplying we get)}$$

k^2+2k=8 or k^2+2k-8=0

applying the quadratic formula we get

$$k= \frac{-2 \pm \sqrt{2^2 - 4(1)(-8)}}{} \quad ->$$

```
              2(1)

  k=2, or k=-4   <--- values of k required to make lines parallel.
```

-----

b)

Lines are perpendicular implies ->

```
   3-1      k-0
  ------ * -----  = -1  ->
  k-(-2)    5-1
   2     k
  --- * ---  = -1  ->
  k+2    4


    2k
  ------ = -1  ->
  4(k+2)


    k
  ------ = -1
  2(k+2)


  -k = 2(k+2) ->

  -k=2k+4 or -4=3k  or  k = -4/3  <--- the value of k required for
                                       lines to be perpendicular.
```


1.10) Find the equation of the line, that is perpendicular to line y=3x+2 and intersects it at x=-1.

By inspection, the line y=3x+2 has a slope of 3, therefore any line perpendicular to it has a slope of -1/3.

next we find the point on line y=3x+2, where x=-1. We substitute -1 in for x into this equation giving us y=3(-1)+2 = -1. Therefore the point on this line where x=1, ... y=1 also. Therefore the point on this line where x=-1 is (-1,-1).

A equation of the line with slope of -1/3 going through the point (-1,-1) is y-(-1)=(-1/3){x-(-1)} which simplifies to

y+1=(-1/3){x+1} or y=(-1/3)x - 1/3 -1 or y=(-1/3)x - 4/3 <---- The equation of the line that is perpendicular to line y=3x+2, and intersects it at x=-1.

1.11) Two slopes are perpendicular and one of them can be represented by the expression –x+3 and the other one can be represented by the expression 9x-1, where x is some yet unknown value. What are each of these slopes? (If there exists two sets of slopes, find both).

The fact that these two slopes are perpendicular indicates that the product of these slopes equals -1. So ...

(-x+3)(9x-1)=-1 ->

-9x^2 + x + 27x -3 = -1 ->

-9x^2+28x-2=0

The quadratic formula is used to solve for x

$$x = \frac{-28 \pm \sqrt{28^2 - 4(-9)(-2)}}{-18} = \frac{-14 \pm \sqrt{178}}{-9} \quad \text{so}$$

$$x = \frac{-14 + \sqrt{178}}{-9} \quad \text{or} \quad x = \frac{-14 - \sqrt{178}}{-9}$$

so there are two sets of slopes which are perpendicular

the first set is

$$-\left[\frac{-14 + \sqrt{178}}{-9}\right] + 3 \text{ and } 9\left[\frac{-14 + \sqrt{178}}{-9}\right] - 1$$

the second set is

$$-\left[\frac{-14 - \sqrt{178}}{-9}\right] + 3 \text{ and } 9\left[\frac{-14 - \sqrt{178}}{-9}\right] - 1$$

Sometimes it is useful to put answers in decimal form. When answers are complex like these are, putting answers in decimal form is a good way to get a feel what the values of the answers really are,

it also
makes it easier to check if the answers are correct. (Because answers in decimal form are frequently not exact, such a check will not tell you with 100% certainty if the answer is correct or not, however in this this case, if this check works, we will be pretty darn certain).

The first set perpendicular slopes are   -0.037963 and 26.3417
The second set of perpendicular slopes are -0.037963 and 26.3417
From this we see that we (probably) don't have two sets of solutions. I say probably because, these decimal answers are not exact, so no conclusion derived from them can be 100% certain however I am quite certain that there is only one set of answers for this problem.

It is possible to show these two sets of answers are the same without converting the answers to decimal form, but it would have been a lot more work. (try it if you wish).

Here we check to see if our answer is (probably) correct.

-0.037963 * 26.3417 = -1.0000099571   close enough, these answers are (probably) correct.

__________________________.

2.T) Derive the 2 dimensional distance formula:

i.e. given the points p1(x1,y1) and p2(x2,y2), the distance between p1 and p1 is sqrt{(x2-x1)^2+(y2-y1)^2};

The picture
Put p1(x1,y1) in the first quadrant. To the right and higher than p1(x1,y1), put p2(x2,y2). The way this is drawn indicates that x2>x1 and y2>y1. Keep in mind this is not necessarily true.

Introduce the point p3(x2,y1) which is directly to the right of P1 and directly below P2. We note segment p1 p3 is parallel to the x axis because the segment's end points share the same y coordinate. We note the segment p2 p3 is parallel to the y axis because the segment's end points share the same x coordinates. Therefore the segments p1 p3 and p2 p3 are perpendicular. This means triangle p1 p2 p3 is a right triangle, where angle p1 p3 p2 is the right angle. The length of segment p1 p3 is |x2-x1|. The length of segment p2 p3 is |y2-y1|. The length of segment p1 p2 is the distance between p1 and p2 and we refer to this distance as d.

The Pythagorean Theorem justifies the following equation.

(p1 p2)^2 = (p1 p3)^2 + (p2 p3)^2 implying

d^2 = (|x2-x1|)^2 + (|y2-y1|)^2 implying

d^2 = (x2-x1)^2 + (y2-y1)^2 implying

d = sqrt{ (x2-x1)^2 + (y2-y1)^2 }

Note: in this context, only positive distances are accepted or allowed, therefore there is no +/- in front of sqrt{ (x2-x1)^2 + (y2-y1)^2 }

Proof Complete

2.1) Use the distance formula to find the distance between the following points. a) (0,0) and (3,4); b) (1,2) and (6,14); c) (5,-1) and (-3,5).

The distance formula is sqrt{(x2-x1)^2+(y2-y1)^2}

We calculate the distance formula between the two points P1(x1,y1) and P2(x2,y2) by plugging the coordinates of these two points into the distance formula, which results in sqrt{(x2-x1)^2+(y2-y1)^2}. This is how to calculate the distance between points using the distance formula. Now this is done for the points of this problem, see below.

a) distance between the points P1(0,0) and P2(3,4) =
   sqrt{(3-0)^2+(4-0)^2} = sqrt(9+16) = sqrt(25) = 5.

b) distance between the points P1(1,2) and P2(6,14) =
   sqrt{(6-1)^2+(14-2)^2} = sqrt{5^2+12^2} = sqrt(25+144) = 13

c) distance between the points P1(5,-1) and P2(-3,5) =
   sqrt[(-3-5)^2+{5-(-1)}^2] = sqrt{(-8)^2+(6)^2} = sqrt(64+36) = 10

2.4) a) Find the distance between the point (1,7) and the line y=3x;
b) Find the distance between the point (a,b) and the line y=cx+d.

The distance between a point between any two objects is the length of the shortest path between the two objects. The shortest path between a point P and a line L is the segment perpendicular to the line, where one endpoint of the segment is P and the other endpoint of the segment is on the line L.

First we find the line L2 perpendicular to y=3x that contains the point (1,7). The slope of such a line is -1/3, utilizing the point slope form of a line, we see this line is y-7=(-1/3)(x-1) which simplifies to y=(-1/3)x+22/3. Next we need to find there the lines y=3x and y=(-1/3)x+22/3 intersect. Substituting 3x in for y in the equation y=(-1/3)x+22/3 gives 3x=(-1/3)x+22/3. We multiply both sides of this equation by 3 to get rid of fractions, which gives

9x=-x+22 -> 10x=22 -> x=22/10. Substituting this value of x into the equation y=3x gives, y=3(22/10) -> y=66/10. Therefore where these two lines intersect is the point (22/10,66/10). So the distance between y=3x and the point (1,7) is the length of the segment (1,7)(22/10,66/10)= sqrt{(22/10-1)^2+(66/10-7)^2} = 4*sqrt(10)/10.

2.4b) The answer to this is not given here. To get this answer, learn to do part a, doing part b is the same idea, only variables are used.

2.7) R is the ray (0,0)(2,6) whose end point is the origin. a) Without making use of the distance formula, propose a point on R that is twice as far from the origin as (2,6), then make use of the distance formula to check your answer. b) Without making use of the distance formula, propose a point on R that is half as far from the origin as (2,6). c) Using the distance formula, determine the distance of (2,6) from the origin, then using the same method used in a and b propose a point on R that is a distance of 1 from the origin.
d) Propose a point on R that is a distance 7 from the origin.

R is the ray (0,0)(2,6) whose end point is the origin. a) Without making use of the distance formula, propose a point on R that is twice as far from the origin as (2,6), then make use of the distance formula to check your answer.

a) the point 2(2,6)=(4,12) is proposed as the point on ray R that is twice as far from the origin as the point (2,6). [Triangle similarity justifies this, it will be left to the student to see why this is so] Next we verify this using the distance formula. The distance of (2,6) from the origin is sqrt(2^2+6^2)=sqrt(40)=
2sqrt(10). The distance of (4,12) from the origin is sqrt(4^2+12^2)= sqrt(16+144)=sqrt(160)=4sqrt(10). Therefore (4,12) is twice as far from the origin as (2,6). Next we show that (2,6) is on the same ray as (4,12). Using the two point form of an equation of a line, we know the equation of the line (0,0)(2,6) is

```
     6-0
y-0= ----(x-0) -> y=3x  likewise, the equation of the line (0,0)(4,12)
     2-0
```

```
        12-0
is y-0= ----(x-0) -> y=3x.  We see that both sets of points lie on
        2,6
```

the same on line, this combined with the fact that (2,6) and (4,12) on the same side of the origin, means both these points lie on the same ray.

. b) Without making use of the distance
formula, propose a point on R that is half as far from the origin

as (2,6).

We multiply both coordinates of (2,6) by 1/2 which gives
(1,3) <--- answer

c) Using the distance formula, determine the distance of (2,6) from the origin, then using the same method used in a and b determine the point on R that is a distance of 1 from the origin.

The distance of (2,6) from the origin is sqrt(2^2+6^2) = 2sqrt(10).

Therefore we expect that the point on ray R that is a distance of 1 from the origin would be gotten by dividing both coordinates of (2,6) by 2sqrt(10). Doing this gives

$$\left[\frac{2}{\text{sqrt}(10)} , \frac{6}{\text{sqrt}(10)}\right] \quad \text{<---- answer}$$

d) Propose a point on R that is a distance 7 from the origin.

Given that the point calculated in part c is a distance of 1 from the origin, it is proposed that multiplying both of its coordinates by 7 would give a point on R that is a distance of 7 from the origin. This point is

$$7 * \left[\frac{2}{\text{sqrt}(10)} , \frac{6}{\text{sqrt}(10)}\right] = \left[\frac{14}{\text{sqrt}(10)} , \frac{42}{\text{sqrt}(10)}\right] \quad \text{<---- answer}$$

2.10) The following sets of points are vertices of rhombus given in counter clockwise order, starting with the lower left vertex, then lower right, then upper right, then upper left. Make use of the distance formula to determine all vertices. Do not make use of vectors when doing this problem. In Appendix 2 vectors will be studied, this problem is easier to do when using vectors. a) (0,0)(1.9225,y2)(3.1261,2.1486)(x4,1.5973); b) (3,2)(8,2)(11,6)(x,y).

Begin by (approximately) drawing this rhombus on a Cartesian coordinate system. Put the (incomplete) coordinate next to each vertex as given above. Since this is a rhombus, each of the sides are the same length, applying the distance formula to the first three coordinates of the rhombus (0,0)(1.9225,y2) and (3.1261,2.1486) leads to the following equation.

sqrt{(1.9225-0)^2+(y2-0)^2} = sqrt{(3.1261-1.9225)^2+(2.1486-y2)^2}

Our goal is to solve for y2. We square both sides of the above equation and don't (for now) consider the negative roots. We discard them. If when we get done, we have an answer that makes sense, we will quit there, if not we will then consider the negative roots.

1.9225^2 + y2^2 = (3.1262-1.9225)^2 + (2.1486-y2)^2 ->

1.9225^2 + y2^2 = 1.2037^2 + 2.1486^2 - 2*2.1486*y2 + y2^2 ->

y2^2 + 3.6960 = y2^2 - 4.2972 y2 + 6.0654 ->

3.6960 = -4.2972 y2 + 6.0654 ->

y2 = (6.0654-3.6960)/ 4.2972 ->

y2 = 0.5513 <---- answer

Now our goal is to solve for x4, we do this in the same way we solved for y2. To do this we consider the following vertexes of the rhombus, (0,0)(x4,1.5973)(3.1261,2.1486) The fact this is a rhombus meaning all of sides have the same length leads to the following equation.

sqrt{(x4-0)^2+(1.5973-0)^2} = sqrt{(3.1261-x4)^2+(2.1486-1.5973)^2} ->

Once again we square both sides of the above equation and don't (for now) consider the negative roots. We discard them. If when we get done, we have an answer that makes sense, we will quit there, if not we will then consider the negative roots.

(x4-0)^2+(1.5973-0)^2 = (3.1261-x4)^2+(2.1486-1.5973)^2 ->

x4^2 + 1.5973^2 = 3.1261^2 - 2*3.1261*x4 + x4^2 + 0.5513^2 ->

1.5973^2 = 3.1261^2 - 2*3.1261*x4 + 0.5513^2 ->

2.5514 = 9.7725 - 6.2522 x4 + 0.30393 ->

-6.2522 x4 = 2.5514 - 9.7725 - 0.30393 ->

x4 = 1.2036 <---- answer

Therefore y2 = 0.5513 and x4 = 1.2036 <---- answer

2.11) Derive 3D Distance Formula
Show that the distance between P(x1,y1,z1) and Q(x2,y2,z2) is given by the formula d = sqrt{(x2-x1)^2+(y2-y1)^2+(z2-z1)^2}.

Consider the points P(x1,y1,z1) and Q'(x2,y2,z1). Since both of these two points have the same z coordinate, they both lie on the plane z=z1 and we can make use of the 2 dimensional distance formula to calculate the distance between these two points. In the z=z1 plane, the coordinates of these two points is (x1,y1) and (x2,y2). The distance between these two points is sqrt{(x2-x1)^2+(y2-y1)^2}.

The distance between Q'(x2,y2,z1) and Q(x2,y2,z2) is |z2-z1|. The points P(x1,y1,z1), Q'(x2,y2,z1), Q(x2,y2,z2) are vertices of a right triangle, with Q' being the vertex of the right angle. Therefore the -distance between P1 and P2- squared equals -the distance between P1 and P2'- squared plus the distance between P2' and P2. Stated mathematically this is

(P1-P2)^2 = (P1-P2')^2 + (P2'-P2)^2 ->

(P1-P2)^2 = [sqrt{(x2-x1)^2+(y2-y1)^2}]^2 + |z2-z1|^2 ->

(P1-P2)^2 = (x2-x1)^2+(y2-y1)^2 + (z2-z1)^2 ->

P1-P2 = sqrt{(x2-x1)^2+(y2-y1)^2 + (z2-z1)^2}

Therefore the distance between two points P1(x1,y1,z1) and P2(x2,y2,z2) is sqrt{(x2-x1)^2+(y2-y1)^2 + (z2-z1)^2}

2.15) Prove triangle A(4,-2,3) B(1,0,4) C(7,10,-12) is a right triangle.

To do this, we calculate the lengths of the three sides of this triangle, using the 3 dimensional distance formula, then we test to see if one side squared + another side squared = the third side squared if so by the Pythagorean theorem, this is a right triangle, if not it isn't.

```
length of AB is sqrt{(4-1)^2+(-2-0)^2+(3-4)^2}=
                sqrt(3^2+(-2)^2+(-1)^2)=
                sqrt(9+4+1)=
                sqrt(14)

length of AC is sqrt{(7-4)^2+{10-(-2)}^2+(-12-3)^2}=
                sqrt(3^2+12^2+(-15)^2)=
                sqrt(9+144+225)=
                sqrt(378)

length of BC is sqrt{(7-1)^2+(10-0)^2+(-12-4)^2}=
                sqrt{6^2+10^2+(-16)^2}=
                sqrt(36+100+256)=
                sqrt(392)
```

from this we see that AB^2+AC^2=BC^2 or

```
[sqrt(14)^2]+[sqrt(378)]^2=[sqrt(392)]^2  ->
14+378=392  ->
392=392
```

Since AB^2+AC^2=BC^2, by the Pythagorean theorem, this is a right triangle.

3.1) I) Use the midpoint formula to find the midpoint of the following segments. II) Use the proportional point formula to find the midpoint of each of the following segments. a) (6,0) (10,2); b) (a,b) (c,d).

```
    (6+10 0+2)                (a+c b+d)
Ia) (----,---) = (8,1);  Ib) (---,---)
    ( 2    2 )                ( 2   2 )


IIa) (6,0)+(1/2){(10,2)-(6,0)} = (6,0)+(1/2)(4,2) = (6,0)+(2,1) = (8,1)

                                                                (c-a d-b)
IIb) (a,b)+(1/2){(c,d)-(a,b)} = (a,b)+(1/2)(c-a,d-b) = (a,b)+(---,---) =
                                                                ( 2   2 )
     (2a+c-a 2b+d-b)   (a+c b+d)
     (------,------) = (---,---)
     (   2      2  )   ( 2   2 )
```

3.2) What is the trisection point of the segment (2,-3) (8,9) a) closest to the point (2,-3)? b) closest to the point (8,9)?

```
a) (2,-3)+(1/3){(8,9)-(2,-3)} = (2,-3)+(1/3)(6,12) = (2,-3)+(2,4) =
   (4,1)

b) (2,-3)+(2/3){(8,9)-(2,-3)} = (2,-3)+(2/3)(6,12) = (2,3)+(4,8) =
   (6,11)
```

^^^^^^^^^
^^^^^^^^^

3.6) Given G(-5,8), K(2,a), H(b,1). a) Find a and b so that K will be the midpoint of GH; b) Find a and b so K will be 3/7 of the way from H towards G.

```
  (-5,8)+(b,1)            (-5+b,9)
a) ------------ = (2,a) -> -------- = (2,a) so
        2                     2

  -5+b
  ---- = 2 -> -5+b = 4 -> b = 9;
   2
```

```
8+1    9             9
--- =  - = a  or a = -
 2     2             2
```

b) (b,1)+(3/7){(-5,8)-(b,1)} = (2,a) ->

(b,1)+(3/7)(-5-b,8-1) = (2,a) so

b+(3/7)(-5-b)=2 -> 7b+3(-5-b)=14 -> 7b-15-3b=14 -> 4b-15=14 ->
4b=29 -> b=(29/4)

1+(3/7)(8-1)=a -> 1+(3/7)(7)=a -> 1+3=a -> a=4

^^^^^^^^
^^^^^^^^

3.7) a) A segment has midpoint M(3,-5), and one end point is A(2,-4).
What are the coordinates of the other end point?

```
(2,-4)+(?,??)   (2+?,-4+??)
------------- = ----------- = (3,5)  so
      2              2
```

```
   (2+?,-4+??)
2*  ---------  = 2*(3,5) ->  (2+?,-4+??) = (6,10) so
        2
```

2+?=6 -> ?=6-2 -> ?=4 and

-4+??=10 -> ??=10+4 -> ??=14

^^^^^^^^
^^^^^^^^

3.9) Prove that two of the medians of the triangle with vertices
(-m,0), (m,0), and (0,3m) are perpendicular to each other.

First we graph this triangle, assuming m>0, we see that two vertices of this triangle lie on the x axis, and the other vertex lies on the y axis. Having graphed the triangle, we have a better sense of it.

In choosing the medians, we choose the medians that touch the following segments of the triangle, (-m,0)(0,3m) and (m,0)(0,3m)

```
                                    (-m  3m )     ( m  3m )
The midpoints of these segments are (---,---) and (---,---)
```

```
                                       ( 2   2 )     ( 2   2 )
```

Therefore two medians of this triangle are

```
      (-m  3m )            ( m ,3m )
(m,0) (---,---) and (-m,0) (---,---)
      ( 2   2 )            ( 2   2 )
```

The slope of the 1st is (3m/2 - 0)/(-m/2 - m) = (3m/2)/(-3m/2) = -1

The slope of the 2nd is {(3m/2)-0}/{(m/2) - -m} = (3m/2)/(3m/2) = 1

Since these two slopes multiplied by each other equals -1, these two slopes are perpendicular.

^^^^^^^^^
^^^^^^^^^

3.12) Make use of the 2d distance formula to prove that the 2d midpoint formula gives the point that it claims to give. (see MPF definition)

Proof

Consider the segment A(x1,y1) B(x2,y2) and the point M{(x1+x2)/2,(y1+y2)/2} which is the point given by midpoint formula as the point on segment AB that is equidistant from A and B.

It will be shown that AM=(1/2)AB and that MB=(1/2)AB, assuming this, these two equations together imply that AM=MB which implies that M is equidistant from A and B. These two equations together also imply that AM+MB=AB. This last equation in conjunction with the triangle inequality implies that M is on segment AB. Therefore these two equations are the basis of a proof that the Midpoint formula gives the point it claims to give.

Proof that AM=(1/2)AB

AM =
sqrt([{(x1+x2)/2}-x1]^2 + [{(y1+y2)/2}-y1]^2) =
sqrt{(x2/2-x1/2)^2 + (y2/2-y1/2)^2 } =
(1/2)sqrt{(x2-x1)^2 + (y2-y1)^2 } = (1/2)AB

so AM=(1/2)AB as required

Proof that MB=(1/2)AB

MB =
sqrt([x2-{(x1+x2)/2}]^2 + [y2-{(y1+y2)/2}]^2) =
sqrt{(x2/2-x1/2)^2 + (y2/2-y1/2)^2) =
(1/2)sqrt{(x2-x1)^2 + (y2-y1)^2 } = (1/2)AB

So MB=(1/2)AB   as required

Proof Complete.

^^^^^^^^
^^^^^^^^

3.13) Use the 2d distance formula to prove that the 2d proportional point formula gives the point it claims to give. (see PPF definition)

Proof

Consider the segment A(x1,y1) B(x2,y2) and the point P{(x1+c(x2-x1),y1+c(y1-y1)} which is the point given by proportional point formula as the point on segment AB that is the fraction c (0<c<1) of the way from point A towards point B.

It will be shown that AP=(c)AB and that PB=(1-c)AB, assuming this, these two equations together imply that AP+PB=AB, which in conjunction with the triangle inequality implies that P is on segment AB. Given that P is on segment AB, the first of these two equations implies that P is the point on AB that is the fraction c of the way from A towards B. Therefore these two equations are the basis of a proof that the proportional point formula gives the point it claims to give.

Proof that AP=cAB

AP =
sqrt([{x1+c(x2-x1)}-x1]^2 + [{y1+c(y2-y1)}-y1]^2) =
sqrt[{c(x2-x1)}^2+{c(y2-y1)}^2] = c*sqrt{(x2-x1)^2+(y2-y1)^2} = cAB
so AP=cAB as required

Proof that PB=(1-c)AB

PB =
sqrt([x2-{x1+c(x2-x1)}]^2 + [y2-{y1+c(x2-x1)}]^2) =
sqrt([(x2-x1)-c(x2-x1)]^2 + [(y2-y1)-c(y2-y1)]^2) =
sqrt([(1-c)(x2-x1)]^2 + [(1-c)(x2-x1)]^2) =
(1-c)sqrt{(x2-x1)^2 + (y2-y1)^2}^2 = (1-c)AB
so PB = (1-c)AB as required

Proof complete

^^^^^^^^
^^^^^^^^

3.14) Propose an equation for each of the following, then verify these proposals are correct using the 3d distance formula. a) the 3d midpoint formula. b) the 3d proportional point formula.

The midpoint formula of the segment $(x_1,y_1)(x_2,y_2)$ is $\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$

We guess that the midpoint formula the segment $P_1(x_1,y_1,z_1)$ $P_2(x_2,y_2,z_2)$ might be

$$Pm\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2},\frac{z_1+z_2}{2}\right)$$

If we can prove that distance (P1 Pm) = distance (Pm P2), i) this would prove that Pm is equidistant from P1 and P2, ii) also (by the triangle inequality) this would prove that Pm lies on the segment P1 P2. These two facts i) and ii) would prove (by the definition of the midpoint) that Pm is the midpoint of the segment P1 P2.

Now we establish that

length of P1 Pm squared or |P1 Pm|^2 =
length of Pm P2 squared or |Pm P2|^2

$$|P_1\,Pm|^2 = \left(x_1 - \frac{x_1+x_2}{2}\right)^2 + \left(y_1 - \frac{y_1+y_2}{2}\right)^2 + \left(z_1 - \frac{z_1+z_2}{2}\right)^2$$

$$\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 + \left(\frac{z_1-z_2}{2}\right)^2$$

$$|Pm\,P_2|^2 = \left(\frac{x_1+x_2}{2} - x_2\right)^2 + \left(\frac{y_1+y_2}{2} - y_2\right)^2 + \left(\frac{z_1+z_2}{2} - z_2\right)^2$$

$$\left(\frac{x_1-x_2}{2}\right)^2 + \left(\frac{y_1-y_2}{2}\right)^2 + \left(\frac{z_1-z_2}{2}\right)^2$$

We see that |P1 Pm|^2 = |Pm P2|^2 therefore the theorem is proved.

Proof Complete

^^^^^^^^
^^^^^^^^

3.16) Prove that the four diagonals of an arbitrary rectangular solid are congruent and intersect at a common midpoint.

Let this rectangular solid (RS) be aligned with each of the 3 axis x, y and z. Let the length in the x direction be a, let the length in the y axis be b, let the length in the z direction be c.

Draw a picture of a rectangular solid, and label coordinates of each of its vertices. Then list the segments that are the four diagonals of the rectangular solid as follows.

d1=(0,0,0)(a,b,c) and
d2=(a,0,0)(0,b,c) and
d3=(a,b,0)(0,0,c) and
d4=(0,b,0)(a,0,c)

```
length of d1 = sqrt{(a-0)^2+(b-0)^2+(c-0)^2} = sqrt(a^2+b^2+c^2)
length of d2 = sqrt{(a-0)^2+(0-b)^2+(0-c)^2} = sqrt(a^2+b^2+c^2)
length of d3 = sqrt{(a-0)^2+(b-0)^2+(0-c)^2} = sqrt(a^2+b^2+c^2)
length of d4 = sqrt{(0-a)^2+(b-0)^2+(0-c)^2} = sqrt(a^2+b^2+c^2)
```

```
                  {(0+a) (0+b) (0+c)}   (a b c)
midpoint of d1 = {-----,-----,-----} = (-,-,-)
                  {  2     2     2  }   (2 2 2)


                  {(a+0) (0+b) (0+c)}   (a b c)
midpoint of d2 = {-----,-----,-----} = (-,-,-)
                  {  2     2     2  }   (2 2 2)


                  {(a+0) (b+0) (0+c)}   (a b c)
midpoint of d3 = {-----,-----,-----} = (-,-,-)
                  {  2     2     2  }   (2 2 2)


                  {(0+a) (b+0) (0+c)}   (a b c)
midpoint of d4 = {-----,-----,-----} = (-,-,-)
                  {  2     2     2  }   (2 2 2)
```

From this we see that all four diagonals are equal, and that the midpoints of all the diagonals are the same, meaning that all the diagonals intersect at a common midpoint.

4.1) Locus Problem - Derive Equations of Circles

a) Make use of the distance formula to find the equation of the set of all points that are a distance 5 from the point (2,3).

b) Make use of the distance formula to derive the equation of the set of all points that are a distance r from the point (x1,y1).

c) Can you see how the shifting theorem (see Section 1) is in effect in problem b? Explain.

-----

a) Make use of the distance formula to find the equation of the set of all points that are a distance 5 from the point (2,3).

distance "point" (x,y) to point (2,3) = 5 -> applying distance formula

$(x-2)^2 + (y-3)^2 = 5^2$

$(x-2)^2 + (y-3)^2 = 25$ <--- This is the answer we seek. This is the equation of the set of all points whose distance from the point (2,3) is 5. This is the equation of the circle whose center is (2,3) and whose radius is 5.

-----

b) Make use of the distance formula to derive the equation of the set of all points that are a distance r from the point (x1,y1).

distance from "point" (x,y) to point (x1,y1) = r ->
applying distance formula we get

$(x-x_1)^2 + (y-y_1)^2 = r^2$ <---- This is the answer we seek. This is the equation of the set of all points that are the distance r from the point (x1,y1). This is the equation of the circle whose center is the point (x1,y1) and whose radius is r.

-----

c) Can you see how the shifting theorem (see Section 1) is in effect in problem b? Explain.

Before you view this explanation, you should do a) and b) first. Stop

The equation of a circle whose center is (x_1,y_1) and whose radius is r is

$$(x-x_1)^2+(y-y_1)^2=r^2$$

A circle whose center is the origin, whose radius is r has the equation $x^2 + y^2 = r$. If we wish to shift this circle so that its center is the point (x_1,y_1), we need to shift it to the right a distance of x_1, and to upward a distance y_1. According to the shifting theorem, if we do this we substitute $x-x_1$ into x and $y-y_1$ into y. Doing this we get the circle equation $(x-x_1)^2 + (y-y_1)^2 = r^2$, which is the same equation for this same circle we derived using locus techniques.

4.13) a) What is the equation of the circle centered at (5,1) and tangent to the line $6y+x=2$? b) Do this problem again using a different method.

a) We will make use of the fact that if a circle is tangent to a line, the circle and the line intersect at one and only one point. From the given information we have

$(x-5)^2+(y-1)^2=r^2$ . . . preliminary equation of circle
$6y+x=2$ . . . equation of line

The last equation implies $x=2-6y$, substituting this result into the preliminary equation of the circle we have,

$(2-6y-5)^2+(y-1)^2=r^2$ which simplifies to

$(-3-6y)^2+(y-1)^2=r^2$ which implies

$9+36y+36y^2+y^2-2y+1=r^2$ which simplifies to

$$37y^2+34y+(10-r^2)=0$$

solving for y we get $$y = \frac{-34 \pm \sqrt{34^2-4(37)(10-r^2)}}{2(37)}$$

if the line and the circle are to intersect at only one point, then y can have only one value, for this to happen, the expression inside the square root sign has to equal zero, therefore

$34^2-4(37)(10-r^2)=0$ solving for r^2 and simplifying we get

$$r^2= 81/37$$

substituting this result into the preliminary equation of the circle we get

```
                  81
(x-5)^2+(y-1)^2=----  <-----  answer to part a)
                  37
```

4.13) a) What is the equation of the circle centered at (5,1) and tangent to the line 6y+x=2? b) Do this problem again using a different method.

b) To solve this problem using a second method. We will find the distance between the center of the circle and the line. This distance is the radius of the circle. Having the center of the circle and the radius of the circle, we have all the information needed for the equation of the circle.

To calculate the distance between the point (5,1) and the line 6y+x=2 we do the following.

We determine the slope of 6y+x=2 by putting it into slope intercept form, i.e. y=1/3-x/6. Therefore the slope of the line 6y+x=2 is -1/6. Next we introduce a line through the point (5,1) that is perpendicular to the line 6y+x=2. The slope of this line would need to be the negative reciprocal of -1/6 or 6. The equation of this line then is y-1=6(x-5) or y=6x-29.

Next we find the point (x,y) where the lines 6y+x=2 and y=6x-29 intersect. We substitute the equation y=6x-29 into the equation 6y+x=2. This gives us 6(6x-29)+x=2 or 36x-174+x=2 or 37x=176 or x=176/37. We calculate y by substituting x into the equation y=6x-29, this gives us y=6(176/37)-29 or y=1056/37-(37)(29)/37=1056/37-1073/37 = -17/37,

so these lines intersect at the point (176/37,-17/37). The distance squared between this point and (5,1) is the radius squared of the circle. So the radius of the circle squared is ...

r^2=(5-176/37)^2+{1-(-17/37)}^2 which gives us
r^2=(5*37/37-176/37)^2+(37/37+17/37)^2 which gives us
r^2=(9/37)^2+(54/37)^2 which gives us

r^2= 81/37^2+2916/37^2 = 2997/37^2 and this reduces to

r^2=81/37 leading to the equation of the circle centered at (5,1) being

(x-5)^2+(y-1)^2=81/37 <------answer to part b

Note: Answers part a and b are the same, so we probably did each correctly.

4.21) Locus Problem - Derive Equation of Perpendicular Bisector

a) Make use of the distance formula to find the equation of the set of all points that are equidistant from the two points (1,2) (-1,7).
b) Calculate the perpendicular bisector of the segment (1,2) (-1,7).
c) Answers a and b should be the same, why?

We let the point (x,y) represent all points that are equidistant from the points (1,2) and (-1,7), applying the distance formula, we express this equi-distance as follows,

sqrt[(x-1)^2+(y-2)^2] = sqrt[{x-(-1)}^2+(y-7)^2]

squaring both sides and remembering that distance is always positive so that we may neglect negative roots we have

(x-1)^2+(y-2)^2 = (x+1)^2+(y-7)^2 ->

x^2-2x+1 + y^2-4y+4 = x^2+2x+1 + y^2-14y+49 ->

canceling the x^2 and y^2 from both sides we have

-2x+1 -4y+4 = 2x+1 -14y+49 ->

14y-4y=2x+2x+49-4 ->

10y=4x+45 ->

```
y=(2/5)x+(9/2)   <---- This is the set of points equidistant from
                        (1,2)& (-1,7)
```

We find the perpendicular bisector of segment (1,2)(-1,7) as follows.

```
                                       (1-1, 2+7)
The midpoint of segment (1,2)(-1,7) is  ---  --- = (0,9/2)
                                         2    2
```

The slope of segment (1,2)(-1,7) is (7-2)/(-1-1)= 5/-2.

The perpendicular bisector of segment (1,2)(-1,7) is perpendicular to this segment and goes through its midpoint. Therefore the slope of the perpendicular bisector is .. the negative reciprocal of -5/2 .. or 2/5. Making use of the point slope form of a line, we know therefore a line equation of the perpendicular bisector is

y-9/2 = 2/5(x-0) -> y-9/2 = (2/5)x -> y=2/5x+9/2.

4.22) Locus Problem - Deriving Equation of Parabola
a) Find the equation of the set of all points that are equidistant

from the line y=-1 and the point (0,1). b) Find the equation of the set of all points equidistant from the line y=-k and the point (0,k). c) What is the vertex and the axis of these parabolas?
Hint: See problem 1.

We represent points that are equidistant from y=-1 and (0,1) as (x,y). The distance from (x,y) to the point (0,1) is sqrt[(x-0)^2+(y-1)^2] = sqrt[x^2+(y-1)^2]. The distance from the point (x,y) and the line y=-1, is y-(-1)=y+1. Mathematically we represent these two distances as being equal by the following equation.

y+1= sqrt[x^2+(y-1)^2] <- this is the set of all points that we seek

we now simplify the above equation, keeping in mind that there are no negative distances, therefore we may dispose of all negative roots.

(y+1)^2 = x^2+(y-1)^2 ->

y^2+2y+1 = x^2 + y^2-2y+1 ->

2y+1 = x^2-2y+1 ->

4y = x^2 ->

```
    x^2
y= -----   <---- this is the equation of the set of points we seek
     4
```

Line Problems

5.1) A man stands 24m from the base of a flag pole, the slope from the man's feet to the top of the flag pole is 1.23, a) How high is the top of the top of the flag pole? b) How far are the man's feet from the top of the flag pole?

a) Assume the man's feet are at the origin. Since the slope of the line from the man's feet to the top of the flag pole is 1.23. The top of the flag pole lies on the line y=1.23x. Assume the base of the flag pole is at the point (24,0). Therefore the entire flag pole lies on the line x=24. Therefore the top of the flag pole lies on the line x=24 Therefore the top of the flag pole is where the lines y=1.23x and x=24 intersect. To find the y coordinate of the top of the flag pole, we substitute the 24 of x=24, into the x of y=1.23x. Therefore y=1.23(24) implying y=29.52. Therefore the top of the flag pole is 29.52 meters tall.

b) Since the entire flag pole lies on the line x=24 and since the flag pole is 29.52 meters tall, the top of the flag pole has coordinates

(24,29.52). Applying the distance formula, the distance of this point to the man's feet (the origin) is sqrt(24^2 + 29.52^2)^2 = 38.045 meters.

6.6) a) Prove: The diagonals of all parallelograms bisect each other.
b) Solve this problem using a different method.

The first solution presented here is the least laborious and illustrates a new trick in solving problem. The outline of a second method is presented here. Solving the problem using this second method will provide the student with excellent experience in algebra. It is best the student not read what experience this is until after they have done this problem on their own. Therefore the benefits garnered will be commented on after the outline of the outline is given.

6.6a

The midpoint of a segment is that point of the segment which is equidistant from the segment endpoints. It was previously proven that the midpoint formula applied to a segment, gives the midpoint of the segment. Therefore if we can show via the midpoint formula, that the diagonals of all parallelograms share a common midpoint we have proven that the diagonals of all parallelograms bisect each other.

The coordinates (0,0)(a,b)(c,0)(a+c,b) where a,b and c are positive, define a general parallelogram. (A general parallelogram is representative of all parallelograms). The diagonals of this parallelogram are the segments S1(0,0)(a+c,b) and S2(a,b)(c,0). The midpoint of S1 is {(a+c)/2,b/2}. The midpoint of S2 is {(a+c)/2,b/2}. The midpoints of these diagonals are the same point, therefore these segments bisect each other and we have proven what we set out to prove.

6.6b

The outline of the second method of solving this problem follows.

1. Give a set of coordinates for a general parallelogram.

2. Find the equations of the lines that contain the diagonals of the general parallelogram.

3. Calculate P, the point of intersection of the diagonals.
4. Verify that P is equidistant from the parallelogram vertices.

Stop: You will get more out of doing this problem of you refrain from reading the following note until after you successfully solve the the problem.

Note: At first it appears doing this problem this way will be quite difficult because at first the formulation of P appears to be overly complex. However if the student will take the time to simplify this formulation, the complexity collapses and the problem becomes much simpler. It is best the student finds this out on their own if they can, if not it is good they discover this, by reading this note.

6.7) Prove: The diagonals of all rhombuses are perpendicular.

(0,0)(a,b)(a+sqrt{a^2+b^2},b)(a+c,0) is a general rhombus (a>0 and b>0). (If you are wondering how this is a rhombus, keep in mind that if a quadrilateral has two opposite sides that are parallel and congruent, it is a parallelogram, also opposite sides of a parallelogram are congruent. You may also verify this by use of the distance formula).

If the slope of the diagonal (0,0)(a+c,b) multiplied by the slope of the slope of the diagonal (a,b)(sqrt{a^2+b^2},0) equals -1, the diagonals are perpendicular.

slope of diagonal (0,0)(a+sqrt{a^2+b^2},b) *
slope of diagonal (a,b)(sqrt{a^2+b^2},0)=

```
        b                   b
---------------- * ---------------- =
 a+sqrt(a^2+b^2)     a-sqrt(a^2+b^2)


         b^2
----------------------- =
a^2 - {sqrt(a^2+b^2)}^2


        b^2
----------------- =
a^2 - (a^2 + b^2)


 b^2
----- =
-b^2

-1
```

Therefore the diagonals are perpendicular, Proof complete.

6.8) Prove: If the diagonals of any quadrilateral are perpendicular and bisect each other, then the quadrilateral is a rhombus.

(a,0)(0,b)(-a,0)(0,-b) is a general quadrilateral whose diagonals bisect each other and are perpendicular. Below the lengths of each of the sides of this quadrilateral are given.

length of side (a,0)(0,b)= sqrt({a-0}^2+{0-b}^2)=sqrt(a^2+b^2)
length of side (0,b)(-a,0)= sqrt({0- -a)^2+{b-0}^2=sqrt(a^2+b^2)
length of side (-a,0)(0,-b)= sqrt({-a-0}^2+{0- -b}^2=sqrt(a^2+b^2)
length of side (0,-b)(a,0)= sqrt({0-a}^2+{-b-0}^2)=sqrt(a^2+b^2)

We see that all sides of this quadrilateral are congruent, therefore this quadrilateral is a rhombus.

Proof Complete

6.28) Prove: (Vertical lines excepted) Two lines are parallel if and only if they have the same slope.

Any two non vertical lines can be represented as

y=mx+a and y=nx+b

calculating where these lines intersect we have

mx+a = nx+b   implying

$$x = \frac{b-a}{m-n}$$

substituting this into y=mx+a, we have $y=m \frac{b-a}{m-n} +a$

so the point of intersection between these two lines is

$$\left[\frac{b-a}{m-n} , m \frac{b-a}{m-n} \right]$$

If the slopes of these two lines are equal, in otherwords if m=n, the value the coordinates of this point is undefined, meaning neither coordinate is equal to a real number, Implying when m=n, the lines do not intersect (anywhere on the coordinate plane) which by definition means these lines are parallel.

If m!=n (m does not equal to n) then both coordinates of the point of intersection of these two lines is a real number. Therefore these points do intersect on the coordinate plane, meaning these two lines are not then parallel.

Therefore two lines are parallel if and only if they have the same slopes.

6.42) Locus Problem - There is an alternate proportional point formula (PPF) which finds the coordinate(s) of the point that divides a segment into 2 parts, such that the ratio of the lengths of the two resulting segments is a given constant. a) Given a segment AB, in one dimension, [Assume the segment AB lies on the number line and that the location of each of its endpoints are specified by a single number]. Using the locus technique derive a PPF that finds a point p, such that Ap divided by pB is the constant c. b) Make use of this PPF to derive the standard PPF previously derived in this book, i.e. p=A+c(B-A)(this PPF finds the point p that divides a segment into two parts, such that the ratio between the segments Ap and the entire segment is a given constant]. c) Now to practice what has been learned, use the locus technique to derive the standard PPF, then make use of the standard PPF to derive the alternate PPF.

a) Given a segment AB, in one dimension, derive a formula to find the point p, such that Ap divided by pB is the constant c.

Ap/pB=c -> [there are two cases to consider to arrive at the following] [equation, i.e. B>A and A>B, both of these assumptions lead] [to the same following result].

$$\frac{p-A}{B-p} = c \quad ->$$

$$p-A = c(B-p) \quad ->$$

$$p-A = cB-cp \quad ->$$

$$p+cp = cB+A \quad ->$$

$$p(1+c) = A+cB \quad ->$$

$$p= \frac{A+cB}{1+c}$$ <---- Answer to a)
This is a proportional point formula which identifies the point p on a segment AB (B>A) for which Ap/pB = c.

b) Make use of this proportional point formula to derive our standard proportional point formula - in one dimension.

Let AB be a segment, let p be the point on this segment, such that Ap/AB equals the constant c'. In other words c'=Ap/(Ap+pB). (Notice that c' is a fixed constant, and as a consequence, p is a fixed point on the segment AB, in other words p is now fixed and defined).

Given that Ap/(Ap+pB) [or Ap/(AB)] equals the constant c', Ap/Bp has a certain value. Once we find this value (or expression) for Ap/Bp, we plug this into the c of the proportional point formula derived in a). The resulting formula will be the proportional point formula we desire i.e. the proportional point formula that finds the point p such that Ap/(AB)=c'.

if Ap/(Ap+pB)=c' what does Ap/pB equal?

c' = Ap/(Ap+pB) ->

Ap= c'(Ap+pB) ->

Ap = c'*Ap + c'*pB ->

c'*pB = Ap-c'*Ap ->

c'*pB = Ap(1-c') ->

Ap/pB = c'/(1-c')

Our new proportional point formula is now derived by plugging c'/(1-c') into c of the proportional point formula derived in a), i.e. ..

$$p = \frac{A+cB}{1+c} \;\rightarrow$$

$$p = \frac{A + \frac{c'}{1-c'} * B}{1 + \frac{c'}{1-c'}} \;\rightarrow$$

$$p = \frac{\frac{A(1-c') + c'B}{1-c'}}{\frac{1-c' + c'}{\;}} \;\rightarrow$$

```
         1-c'


p= A(1-c')+c'B  ->

p= A-Ac'+c'B + ->

p= A+c'(B-A)
```

Up to this point, we had the variables c' and c, we kept these variables distinct so they wouldn't be confused with each other. From here on we refer to the variable c' as c, so our new proportional point formula is

```
p=A+c(B-A)  <---  Answer to b)
                 This is the proportional point formula which identifies
                 the coordinate of the point p on the segment AB such
                 that Ac/AB=c.
```

This is the formula we expected, we have derived this particular point formula before in this book in the "Midpoint and Proportional Point Formula Section".

7.12) Alternate Proportional Point Formula (PPF) - Derivation

a) Given two points P1(x1,y1) and P2(x2,y2) let p be the point on the segment P1_P2 (or the line P1_P2) such that the (the length of the segment) P1_p divided by P2_p equals c. Derive a formula which gives the coordinates of c. b) Make use of this alternate form of the PPF to derive the more standard form of the PPF given in this book previously i.e. where p is the point such that c equals P1_p divided by P1_P2. This proportional point formula in vector form is p=P1+c(P2-P1).

In solving this problem, assume that the point p lies on the line that contains P1 and P2. This assumption makes use of constraint(s) which is what should be done with all problems in this section. This assumption also makes this proof easier to do, since the assumption doesn't have to be proved, as we did in prior derivations of the PPF.

The equation of the line that passes through the points P1(x1,y1) and P2(x2,y2) is

```
     y2-y1
1) y=-----(x-x1)+y1
```

```
     x2-x1
```

an other obvious equation form of this line is

```
     y2-y1
2) y=-----(x-x2)+y2
     x2-x1
```

equation 1 implies that the point p is constrained lie on the general point

```
   y2-y1
{x,-----(x-x1)+y1}
   x2-x1
```

this same general point in a different form (related to equation 2) is

```
   y2-y1
{x,-----(x-x2)+y2}
   x2-x1
```

Given that c is the ratio it is defined to be in the problem we have

```
   P1_p
c=------  ->
   p2_p
```

Note: In doing this problem we assume the point p is p(x,y) where (x,y) are the yet unknown (to be calculated) coordinates of p. This problem will be done when we have solved for x and y.

note: The equation above is an expanded form of the equation below. In the equation below, the two different forms of the point that p is constrained to lie on are both used. This is because one of these forms works best when matched with (x1,y1) and the other form works best when matched with (x2,y2). The problem will be easier to solve because of this choice.

```
               y2-y1
    (x1,y1)_(x,-----(x-x1)+y1)
               x2-x1
c=  ---------------------------  ->
               y2-y1
    (x2,y2)_(x,-----(x-x2)+y2)
               x2-x1
```

```
                     y2-y2
      (x1-x)^2+(y1 - -----(x-x1)-y1)^2
```

```
                     x2-x1
c^2= -----------------------------------  ->
                      y2-y1
      (x2-x)^2+(y2 -  -----(x-x2)-y2)^2
                      x2-x1



                y2-y2
      (x1-x)^2+{-----(x-x1)}^2
                x2-x1
c^2= ---------------------------  ->
                y2-y1
      (x2-x)^2+{-----(x-x2)}^2
                x2-x1


        (x1-x)^2 (x2-x1)^2 + (y2-y1)^2 (x1-x)^2
        ----------------------------------------
                        (x2-x1)^2
c^2= ----------------------------------------------  ->
        (x2-x)^2 (x2-x1)^2 + (y2-y1)^2 (x2-x)^2
        ----------------------------------------
                        (x2-x1)^2




        (x1-x)^2 (x2-x1)^2 + (y2-y1)^2 (x1-x)^2
c^2= ----------------------------------------------  ->
        (x2-x)^2 (x2-x1)^2 + (y2-y1)^2 (x2-x)^2


       (x1-x)^2 [(x2-x1)^2 + (y2-y1)^2]
c^2=  ----------------------------------  ->
       (x2-x)^2 [(x2-x1)^2 + (y2-y1)^2]


      (x1-x)^2
c^2=  --------  ->
      (x2-x)^2


         (x1-x)
(+/-)c= --------  ->
         (x2-x)

(+/-)c (x2-x) = (x1-x)
```

$$(+/-)c * x2 - (+/-)c * x = x1-x \;\; ->$$

$$-(+/-)c * x + x = -(+/-)c * x2 + x1 \;\; ->$$

$$x-(+/-)c*x = x1-(+/-)c * x2 \;\; ->$$

$$x[1-(+/-)c] = x1-(+/-)c * x2 \;\; ->$$

$$x = \frac{x1-(+/-)c * x2}{1-(+/-)c} \;\; ->$$

The variables (+/-)c in the numerator and in the denominator are the same variable because they both come from a single (+/-)c previously therefore they both have to be positive or they both have to be negative therefore this all leads to

$$x = \frac{x1-cx2}{1-c} \quad \text{or} \quad x = \frac{x1+cx2}{1+c}$$

both of these results for x satisfy that x1_p / x2_p = c. However only one of these satisfies that p is on the segment (not line) x1 x2. We know from how p is defined that if c=1, then the x coordinate of p must be (x1+x2)/2. Only the second of these choices satisfies this criteria, therefore

$$x = \frac{x1+cx2}{1+c} \quad \text{<----} \quad \text{This is the x coordinate of p}$$

substituting this value of x into the equation $y=\frac{y2-y1}{x2-x1}(x-x1)+y1$ implies

$$y=\frac{y2-y1}{x2-x1}\left[\frac{x1+cx2}{1+c} - x1 \right] + y1 \;\; ->$$

$$y=\frac{y2-y1}{x2-x1}\left[\frac{x1+cx2 - x1(1+c)}{1+c} \right] + y1 \;\; ->$$

$$y=\frac{y2-y1}{x2-x1}\left[\frac{cx2 - cx1}{1+c} \right] \;\; ->$$

$$y = \frac{y_2-y_1}{x_2-x_1} * c * \frac{x_2-x_1}{1+c} + y_1 \rightarrow$$

$$y = \frac{c(y_2-y_1)}{1+c} + y_1 \rightarrow$$

$$y = \frac{c(y_2-y_1) + (1+c)y_1}{1+c} \rightarrow$$

$$y = \frac{cy_2-cy_1+y_1+cy_1}{1+c} \rightarrow$$

$$y = \frac{y_1 + cy_2}{1+c} \quad \leftarrow \text{ The y coordinate of p}$$

as expected, the y coordinate of p has the same form as the x coordinate of p. (We know this from experience with the standard PPF).

Therefore

$$p = \left[\frac{x_1+cx_2}{1+c}, \frac{y_1+cy_2}{1+c} \right]$$

to be continued

10.23) Challenging Problems

Radius of Circle Inscribing Triangle
A triangle has sides, s1, s2 and s3. Prove that the radius of the circle that passes through all three vertices of this triangle is r={s1*s2*s3}/{4*area of the triangle}.

Proof:

Let the triangle have vertices of (0,0), (2a,2b), (2c,0). The three vertices define the three sides of the triangle. The intersection of the perpendicular bisector of any two sides is the center of the circle that passes through all three vertices [take time to understand why before moving on]. For simplicity sake we choose to find the intersection of the perpendicular bisectors of the sides s1=(0,0)(2a,2b) and s2=(0,0)(2c,0). pb1 and pb2 are the perpendicular bisectors for s1 and s2 respectively.

finding the line equations for pb1 and pb2
I) pb1 is y-b=(-a/b)(x-a) or pb1 is y=b+(-a/b)(x-a)
II) pb2 is x=c

calculating the intersection of pb1 and pb2
by substituting equation II into equation I we have

III) y=b+(-a/b)(c-a) or y=(a^2+b^2-ac)/b

from equation II we have x=c, so the intersection point of pb1 and pb2 is {c,(a^2+b^2-ac)/b} which is the center of the circle or cc.

cc is equidistant from each of the vertices of the triangle. So the distance from cc to the origin (one of the vertices) is the radius of the circle, so

r= distance from cc to origin, leading to

```
     r^2=c^2+{(a^2+b^2-ac)/b}^2 =
IV)  (a^4+b^4+2a^2*b^2+a^2*c^2+c^2*b^2-2a^3*c-2b^2*a*c)/b^2

 (4*area of triangle)^2=(4*1/2*base*height)^2=(2*s2*2b)^2
V)                     =(2*2c*2b)^2=(8cb)^2=64(c*b)^2

VI)   s1^2=4a^2+4b^2
VII)  s2^2=4c^2
VIII) s3^3=(2a-2c)^2+(2b)^2=4a^2-8ac+4c^2+4b^2

     (s1*s2*s3)^2={(4a^2+4b^2)(4c^2)}(4a^2+4b^2+bc^2-8ac)
               =(16(a*c)^2+16(b*c)^2)(4a^2+4b^2+4c^2-8ac)
IX) = 64a^4*c^2+128a^2*b^2*c^2+64a^2*c^4
     -128a^3*c^3+64b^4*c^2+64b^2*c^4-128a*b^2*c^3

(s1*s2*s3)^2/(4*area triangle)^2   (see equations IX and V)

 =  64a^4*c^2+128a^2*b^2*c^2+64a^2*c^4
  -128a^3*c^3+64b^4*c^2+64b^2*c^4-128a*b^2*c^3
  ----------------------------------------------
```

```
                    64(c*b)^2
```

X) = (a^4+b^4+2a^2*b^2+a^2*c^2+c^2*b^2-2a^3*c-2b^2*a*c)/b^2

we see that equation IV equals equation X. This proves

r^2={s1*s2*s3}^2/{4*area of the triangle}^2

given that areas and distances can not be negative, this proves.

r={s1*s2*s3}/{4*area of the triangle}   proof complete.

____.

10.24) L1 and L2 are parallel lines. L1 is above L2. The distance between them is 1. L1 intersects the y axis at 7 and L2 intersects the x axis at 3. a) What is the slope of L1 and L2?; b) What is the x intercept of L1?; c) What is the y intercept of L2?

1) L1 is y=mx+7
2) L2 is y=mx+?

substituting the point (3,0) into equation 2 we have

3) 0=3m+? -> ?=-3m

substituting equation 3 into equation 2 we have

4) L2 is y=mx-3m

we introduce L3 such that it contains the point (3,0) and is perpendicular to L1 and L2.

5) L3 is y=-1/m(x-3) or y=-1/m*x+3/m

now we find where L1 and L3 intersect.
L1: y=mx+7 and L3: y=-1/m*x+3/m implies mx+7=-1/m*x+3/m
solving for x we have

6) x=(3-7m)/(m^2+1)
substituting equation 6 into equation 1 we have

```
      ( 3-7m  )     ( 3m-7m^2+7m^2+7 )   ( 3m+7  )
7) y=m(-------)+7 = (----------------) = (-------)
      ( m^2+1 )     (     m^2+1      )   ( m^2+1 )
```

therefore from equations 6 and 7, the point of intersection between line 1 and line 3 is

```
                                 ( 3-7m      3m+7  )
8) L1 & L2 intersection = (-------, -------)
                                 ( m^2+1     m^2+1 )
```

Given that L1 and L2 are 1 unit apart and L3 is perpendicular to L1 & L2 we know the distance between (3,0) and (L1 & L2) intersection is 1, therefore,

```
( 3-7m        )2     ( 3m+7  )2
(------- - 3)   + (-------)   = 1   or
( m^2+1       )     ( m^2+1 )


( 3m^2+7m )2     ( 3m+7  )2
( ------- )   + ( ----- )    = 1   or
(  m^2+1  )      ( m^2+1 )


      ( 3m+7  )2     ( 3m+7  )2
m^2 * (-------)   + ( ----- ) = 1   or
      ( m^2+1 )      ( m^2+1 )


( 3m+7  )^2
(-------)   * (m^2+1) = 1   or
( m^2+1 )


( 3m+7  )^2
 ----------   = 1   or
   m^2+1


(3m+7)^2 = m^2+1   or
```

```
                                             -1.68127                   -3.56873
9) 8m^2 + 42m + 48 = 0
                                         -42 + sqrt(228)         -42 - sqrt(228)
   solving for m we get either r1 = ---------------   r2=  --------------
                                               16                     16
```

we have r1 = -1.68127... and r2 = -3.56873...

From equation 4, the y intercept of L2 is y=m(0)-3m or y=-3m. If we assume that m=r2, the y intercept of L2 is -3r2 or 10.706... thus ruling out m=r2 because that would make the y intercept of L2: y=mx-3m greater than the y intercept of L1: y=mx+7. Therefore m=r1.

10) m = r1 = -1.681 270 695 591   <- answer

Calculating the x intercepts of L1: (see equation 1)

L1: y=mx+7
substituting 0 into y we have 0=mx+7 -> x=-7/m = 4.16351752181
11) L1: x intercept = 4.163 517 521 81   <- answer

Calculating the y intercept of L1:
L1: y=mx+7
substituting 0 into x we have y=0m+7 -> y=0
12) L1: y intercept = 0

Calculating the x intercept of L2: (see equation 4)
L2: y=mx-3m
substituting 0 into y we have 0=mx-3m -> x=3
13) L2: x intercept = 3

Calculating the y intercept of L2
L2: y=mx-3m
substituting 0 into x we have y=0m-3m = 5.04381208677
14) L2: y intercept = 5.043 812 086 77 <- answer

____.

10.25) The perimeter of a right triangle is 60 inches and the altitude perpendicular to the hypotenuse is 12 inches. What are the lengths of the sides of the triangle?

The picture
The vertex opposite the hypotenuse is at the origin. The other vertex is is at (0,a). The other vertex is at (b,0).

First we determine the coordinate where the altitude to the hypotenuse intersects the hypotenuse. The slope of the hypotenuse is –a/b, the hypotenuse has a y intercept of a, therefore the line that contains the hypotenuse is,

1) y=a-(a/b)x   the line that contains the hypotenuse

The altitude to the hypotenuse contains the origin and has a slope of b/a therefore the line that contains the altitude to the hypotenuse is

2) y=(b/a)x   the line that contains the altitude to the hypotenuse
Now we find the point where the hypotenuse and the altitude to the hypotenuse meet. So we set equation 1 and 2 equal to each other to find the x coordinate.

3) (b/a)x=a-(a/b)x

solving for x we get

```
        a^2*b
4) x= ---------
```

$$a^2+b^2$$

substituting equation 4 into equation 2, we solve for y

5) $$y= \frac{a*b^2}{a^2+b^2}$$

applying the distance formula to the end points of the altitude to the hypotenuse we obtain

6) $$\left(\frac{a^2*b}{a^2+b^2}\right)^2 + \left(\frac{a*b^2}{a^2+b^2}\right)^2 = 12^2$$

putting both terms in equation 6 over the common denominator, then factoring out a^2*b^2 from the numerator, we cancel an (a^2+b^2) from the numerator and denominator and equation 6 simplifies to

7) $$\frac{a^2*b^2}{a^2+b^2} = 12^2$$

The problem tells us the perimeter of this right triangle is 60, from “the picture” we know one side of the right triangle is a, the other is b therefore

8) $a+b+\mathrm{sqrt}(a^2+b^2) = 60$

Starting from equation 7, we solve for b^2 as follows

$a^2*b^2=12^2*a^2+12^2*b^2$
$a^2*b^2-12^2*b^2=12^2*a^2$
$b^2(a^2-12^2)=12^2*a^2$

9) $$b^2 = \frac{12^2*a^2}{a^2-12^2} = \frac{12^2*a^2}{(a+12)(a-12)}$$

From equations 7 and 8 respectively we solve for $\mathrm{sqrt}(a^2+b^2)$ two different ways and get

10) $\mathrm{sqrt}(a^2+b^2) = ab/12$
11) $\mathrm{sqrt}(a^2+b^2) = 60-a-b$

Setting equations 10 and 11 equal to each other we get

12) $ab/12=60-a-b$
Starting from equation 12 and solving for b we get

```
         720–12a
13)  b= ----------
          a+12
```

setting equation 9 and equation 13 squared equal to each other we get

```
  12^2*a^2      (720–12a)2         12^2*a^2        12^2(60-a)^2
------------ = (-------)   ->  ------------  = --------------  ->
(a+12)(a-12)   ( a+12  )       (a+12)(a-12)     (a+12)(a+12)
```

```
 a^2      (60-a)^2
----- =  --------  -> (a-12)(60-a)^2 = a^2(a+12) ->
a-12      a+12
```

14) a^2-35a+300=0

Using equation 14 to solve for a we get

15) a=15 or a=20

Using equation 15 and equation 13 we get

16) when a = 15, b = 20 and when a=20, b = 15;   <--- Answer

____.

10.28) Challenging Problems

Ladder Problem
Two buildings, A and B stand next to each other forming an alleyway. Two ladders in the alley cross each other touching at the point where they cross. The bottom of one latter sits against the base of building A, and leans over on building B. The bottom of the other ladder sits against the base of building B, and leans over on building A. One ladder is 3 meters long the other ladder is 4 meters long. The point where they cross is 1 meter above the ground. How far apart are the buildings?

The Solution

ladder 1 is 3 meters long and has end points (0,0) and {w,sqrt(9-w^2)}.
ladder 2 is 4 meters long and has end points (w,0) and {0,sqrt(16-w^2)}.

If one starts at the base of ladder 1 and goes some unknown proportion I of the distance towards the top of ladder 1 they will be at the point where the ladders cross.

If one starts at the base of ladder 2 and goes some unknown proportion

II of the distance towards the top of ladder 2 they will be at the point where the ladders cross.

Equation 1 below is in vector proportional point format, it expresses where the ladders cross, and accepts as its inputs the endpoints of the ladders and the height at which they cross.

1) pp(?,1)=(0,0)+I[{w,sqrt(9-w^2)}-(0,0)]
   =(w,0)+II[{0,sqrt(16-w^2)}-(w,0)]

simplifying we get

2) {I*w,I*sqrt(9-w^2)}={w-w*II,II*sqrt(16-w^2)}=(?,1)

acknowledging that y coordinates in equation 2 are equal we get

3) I*sqrt(9-w^2)=II*sqrt(16-w^2)=1

this leads to

4) I=1/sqrt(9-w^2) and

5) II=1/sqrt(16-w^2)

acknowledging that the x coordinates of equation 2 are equal we have

6) I*w=w-w*II or I*w=w(1-II)

dividing through by w we get

7) I=1-II

substituting equations 4 and 5 into equation 7 we have

8) 1/{sqrt(9-w^2)}=1-1/sqrt(16-w^2)

Equation 8 can be manipulated into a 4th degree polynomial and then solved using a quartic equation, however here we solve it as is, numerically. The solve procedure on a HP48 calculator was used to solve equation 8 for w, the answer is ...

9) w=2.60328775442 meters <-------- the solution

10.31) The Scaling Theorem

Theorem Compression / expansion

If any function y=f(x) is compressed horizontally, about the y axis by a factor of k, such that every point of the function becomes k times closer to the y axis, then the function becomes y=f(k*x).

For example, the period of the function y=cos(x) is 360 degrees or 2Pi radians. The function y=cos(k*x) has a period of 360/k degrees or 2Pi/k radians.

Proof

{a,f(a)} is a point of the function y=f(x). [Justification: 'a' substituted into y=f(x) is f(a)]. Therefore the general point {a,f(a)} and the function y=f(x) represent the same set of points. If the set of points {a,f(a)} is compressed towards the y axis such that each point becomes 'k' times closer to the y axis, this set of points becomes {a/k,f(a)}. {a/k,f(a)} is a point of the function y=f(k*x). [Justification: a/k substituted into y=f(k*x) is f(a)]. Therefore the general point {a/k,f(a)} and the function y=f(k*x) represent the same set of points. Therefore if y=f(x) is compressed horizontally about the y axis by a factor of k, such that every point becomes k times closer to the y axis, this function becomes y=f(k*x).

Proof Complete

10.33) Locus Problem: Prove that the set of points that is c (c>1) times as close to one line as to the other line is a set of two lines.

Proof:

In this part of the proof, we assume two lines in a plane that intersect and are not perpendicular. P(a,b) is a set of points such that P is c (c>0) times as close to one of these lines as to the other line. A Cartesian coordinate system can be chosen for these two such that these lines intersect at the origin and so that the line that is closest to the set of points P is aligned with the x axis. Assuming this is so; the equations of these lines are L1:y=0, and L2:y=mx where

The line through (a,b) perpendicular to y=mx is y-b=(-1/m)(x-a) or y=(-1/m)x+a/m+b

seeking to find where these this line and y=mx intersect we set

mx=(-1/m)x+a/m+b

which simplifies to

```
   a+bm
x=-------
   m^2+1
```

substituting this value of x into y=mx gives

```
   m(a+bm)
y=--------
   m^2+1
```

Therefore these two lines intersect at the point

```
(a+bm  m(a+bm)
(-----,------)
(1+m^2 1+m^2 )
```

The distance between this point and (a,b) is the distance between (a,b) and the line y=mx. This distance is ..

```
    { (a+bm     )2   (m(a+bm)  )2 }
sqrt{ (----- - a)  + (------ -b)  } =
    { (1+m^2    )    (1+m^2    )  }


    { (a+bm-a-am^2)2   (ma+bm^2-b-bm^2)2 }
sqrt{ (-----------)  + (--------------)  } =
    { (   1+m^2   )    (     1+m^2    )  }


    { (bm-am^2)2   (ma-b) )2 }
sqrt{ (-------)  + (------)  }
    { ( 1+m^2 )    (1+m^2 )  }
```

This distance between the point (a,b) and the line y=0 is b. The distance between (a,b) and the line y=mx is c times this distance, put another way the distance between the point (a,b) and the line y=mx is cb. Therefore we have the following equation

```
    { (bm-am^2)2   (ma-b) )2 }
sqrt{ (-------)  + (------)  } = cb  ->
    { ( 1+m^2 )    (1+m^2 )  }


(bm-am^2)2   (ma-b )2
(-------)  + (-----)  = (cb)^2  ->
( 1+m^2 )    (1+m^2)


m^2(b-am)^2 + (b-am)^2 = (cb)^2(1+m)^2 ->

(m^2+1)(b-am)^2 = (cb)^2(1+m^2)

(b-am)^2 = (cb)^2
```

$b-am = (\pm)cb$

remembering that a is the x coordinate of the point (a,b) and b is the y coordinate of the point (a,b), this last equation implies

$y-ax = (\pm)cy$

or

$y-ax=cy$ and $y-ax=-cy$ which implies

$y-cy=ax$ and $y+cy=ax$ which implies

$y(1-c)=ax$ and $y(1+c)=ax$ which implies

$$y=\left(\frac{a}{1-c}\right) x \quad \text{and} \quad y=\left(\frac{a}{1+c}\right)$$

This proves that the set of points that is c ($c>0$) times closer to one line as to another line is a set of two lines assuming the lines are not parallel or perpendicular. The to prove these two cases is the remaining part of this proof and has yet to be done. If I don't get around to it it is given as an exercise to the student to do finish this proof.

10.37) All Parabolas are Similar

If two geometrical objects have the same shape, but not necessarily the same size, they are said to be similar. Triangles that have the same angles are all similar. All circles are similar. All squares are similar. Not all rectangles are similar, not all triangles are similar. Prove that all parabolas are similar, i.e. prove that $y=x^2$ is similar to $y=ax^2$. (Hint: See the "Scaling theorem"). Note: Parabolas are the set of points that are equidistant from a line, and a point not on that line. All lines together with a point not on the line are similar regardless of the distance between the line and the point. (Can you see why)? Given this, it seems reasonable that all parabolas should be similar. This is not a proof that all parabolas are similar, but is another way to see why they ought to be similar.

Proof

According to the scaling theorem, with respect to $y=cx^2$, the function $ay=c(ax)^2$ ($a\neq 0$) is compressed about the origin, in a direction parallel to the x axis by a factor of a and is compressed about the origin in a direction parallel to the y axis by a factor of b. Therefore ay=c(ax)^2 is similar to y=cx^2.

Therefore we have the following
(here we use 1/c (c!=0) in place of a)

a) y=cx^2 is similar to $\frac{1}{c}*y=c\left(\frac{1}{c}*x\right)^2$

b) which is the same function as $\frac{1}{c}*y = c*\left(\frac{1}{c}\right)^2 *x^2$

c) which is the same function as y=x^2

Since y=x^2 is the same function as $\frac{1}{c}*y=c\left(\frac{1}{c}*x\right)^2$

(by the transitive law of equality), and since this last function is similar to y=x^2, we have that y=x^2 is similar to y=cx^2.

Proving that all parabolas are similar to each other.
(rotation and translation doesn't affect similarity)
Proof Complete

10.41)

Prove: All three medians of a triangle meet at a common point, and this point is two thirds of the distance from any triangle vertex towards the midpoint of the opposite side.

Proof: (This proof uses vectors and the proportional point formula).

Let ABC be an arbitrary triangle (A,B,C are points). B<->AC is the median of this triangle whose end points are, B and the midpoint of the side opposite of B (segment AC). A<->BC and C<->AB, similarly defined, are the other two medians of the triangle.

Let D be the midpoint of AC, the side opposite B, we have

a) D=(A+C)/2

let m be the point

b) m=B+I{D-B} (where I is yet to be determined constant).

{m is on the median B<->AC for (0<=I<=1)}, this is true because of the properties of the proportional point formula, which were previously proved}.

substituting equation a) into equation b) we have,

c) m=B+I{(A+C/2)-B}

rearranging equation c) we have

d) m=A(I/2)+B(1-I)+C(I/2)

{equation d) gives the location of a point m which is constrained to lie on median B<->AC, (0<=I<=1)}.

Equation d) implies that if a point lies on a median of a triangle then the coefficients of the two vertices that are not one of the medians endpoints are equal. Therefore if a point is to lie on all three medians A<->BC, B<->AC, and C<->AB, the following equalities would apply for equation d), coef B=coef C, coef C=coef A, coef A=coef B meaning coef A=coef B=coef C. Therefore we seek a value of I which will cause the coefficients of A,B and C to be equal, Which we do in the next two equations e) and f).

e) I/2=1-I {refer to equation d)}

solving equation e) for I, we get

f) I=2/3

substituting equation f) into equation d) we get

g) m=A(1/3)+B(1/3)+C(1/3)

Substituting equation f into equation b tells is that m is the point on the median B<->AC that is two thirds of the way from B, (a vertice of triangle ABC) towards the midpoint of AC, (the side opposite B).

We will make use of symmetry to finish this proof. If triangle vertices are renamed (A becomes B, B becomes C, C becomes A for example), then the median that B<->AC represents changes. Through an appropriate renaming B<->AC can be made to represent any of the medians of triangle ABC. However any such renaming doesn’t change the location of the point represented by the equation m=A(1/3)+B(1/3)+C(1/3). This is true because the expression A(1/3)+B(1/3)+C(1/3) is symmetrical* with respect to the variables A,B and C. Therefore m is a single point, regardless of how the triangle vertices are named. Since all medians of triangle ABC can be represented as B<->AC, what the above proof says about the particular median named B<->AC applies to all medians of triangle ABC. Therefore all three medians of the triangle ABC contain THE point m=A(1/3)+B(1/3)+C(1/3) and this point is two thirds of the distance from any triangle vertex towards the midpoint of the opposite side. Therefore medians of triangle ABC do intersect at a common point.

Since ABC can represent any triangle, what we have proven for triangle ABC we have proven for all triangles, that is, all three medians of any triangle meet at a common point, and this point is two thirds of the distance from any triangle vertex towards the midpoint of the opposite side.

* An equation is symmetrical with respect to certain variables if the act of having the variables change places with one another in the equation does not change the equation.

Proof Complete.

Discussion

Allowing B<->AC to represent all medians of the triangle is a use of symmetry. This would not have been possible were it not for shared attributes (symmetry) between the medians. Making use of the fact that the expression A(1/3)+B(1/3)+C(1/3) is symmetrical with respect to the variables A,B,C is also a use of symmetry. These uses of symmetry, allowed the proof lines a through g to be done only one time, not three.

One obvious or straight forward way to have accomplished this proof would be to repeat this proof (lines a through g) three times, once for each of the three medians, verifying that the point m=A(1/3)+B(1/3)+C(1/3) is on all three medians, etc. This would have been more labor intensive. Another way to do this proof would be to use strictly classical coordinate geometry, i.e. finding the equations of the lines that contain the medians of the triangle, and then verifying these three lines intersect at a common point. This would have been much more labor intensive also.

10.42) Napoleon's Theorem 1

If equilateral triangles are placed on all three sides of any arbitrary triangle, such that the arbitrary triangle and each of the equilateral triangles share a common side, and the equilateral triangles are pointed outward with respect to the arbitrary triangle, prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the arbitrary triangle's, Napoleon triangle 1.

Notes:

This proof uses the & operator, to understand its meaning see Appendix 2 vector rotation section.

The following proof relies heavily upon substitutions, these are justified by the symmetry of the situation, and the fact that if the names vertices of the triangle A,B,C substituted for different names via a substitution, an existing proof would prove what it does, for a different set of points. Substitutions and symmetry are powerful tools, that can greatly reduce the work required to do proofs in coordinate geometry.

This proof introduces a tool new to this book so far, having a computer do algebra manipulations. The advantage having a computer do algebra manipulations similar to having a computer find the roots of certain equations such as a 4th order polynomial. In each case, a lot of work is saved. In neither case does the computer take responsibility from the student to devise a logical proof. It is up to the student doing the proof to set up the computer program so that it will do the required task.

Let A(x1,y1), B(x2,y2) and C(x3,y3) be the vertices of a general triangle, such that A-B-C is counterclockwise around the triangle. The following vector equation gives the location of the centroid of the equilateral
triangle pointed outward from triangle ABC, whose side is the segment AB. In short, the following equation gives the centroid associated with AB.

The tools of substitution and the computer for doing algebra keep the this proof from being huge, and from taking an enormous amount of work.

___________________ end notes

Proof

In problem 10.39, it was proven that the centroid of any triangle (where the three medians of the triangle meet) is two thirds of the way from any vertex towards the mid point of the side opposite this vertex. In an equilateral triangle, if the sides are length L, the medians are length L*sqrt(3)/2. Therefore to get to a median, start at the midpoint of any side and go a distance of L*sqrt(3)/6 towards the vertex opposite the side.

Given that A-B-C is counterclockwise around the triangle ABC, equation a) defines the point that is the centroid of the equilateral triangle with one side AB, and pointed outward from triangle ABC. All centroids defined from equations derived from a), are also of triangles pointed outwards triangle ABC.

```
   (A+B)              sqrt(3)
a) ----- + &(B-A) * -------  =
     2                  6
```

```
   (x1,y1)+(x2,y2)                          sqrt(3)
b) --------------- + &{(x2,y2)-(x1,y1)} * ------- =
         2                                     2
   (x1   x2  y1   y2)                     sqrt(3)
c) (-- + --, -- + --) + &(x2-x1,y2-y1) * ------- =
   (2    2   2    2 )                       6

    (x1   x2  y1   y2)                    sqrt(3)
d)  (-- + --, -- + --) + (y1-y2,x2-x1) * ------- =
    (2    2   2    2 )                       6
```

The centroid associated with then AB is (equation e = equation d)

```
    (x1   y1*sqrt(3)   x2   y2*sqrt(3)  -x1*sqrt(3)   y1   x2*sqrt(3)
e)  (-- + ---------- + -- - –--------- , ---------- + -- + ----------
    (2        6        2        6            6        2        6

      y2)
    + --)
      2 )
```

Making the following substitutions A->B, B->C, C->A, we find
The centroid associated with BC is

```
   (x2   y2*sqrt(3)   x3   y3*sqrt(3)  -x2*sqrt(3)   y2   x3*sqrt(3)
f) (-- + ---------- + -- - –--------- , ---------- + -- + ----------
   (2        6        2        6            6        2        6

     y3)
   + --)
     2 )
```

Making the following substitutions B->C, C->A, A->B, we find
The centroid associated with CA is

```
   (x3   y3*sqrt(3)   x1   y1*sqrt(3)  -x3*sqrt(3)   y3   x1*sqrt(3)
g) (-- + ---------- + -- - –--------- , ---------- + -- + ----------
   (2        6        2        6            6        2        6

     y1)
   + --)
     2 )
```

We give the centroid associated with each of the following sides names as follows.

h) A’(xa,ya) is the centroid associated with AB.
i) B’(xb,yb) is the centroid associated with BC
j) C’(xc,yc) is the centroid associated with CA

Now we prove that A′ lies on the perpendicular bisector of the segment B′C′

k) The midpoint of A′B′ = $\left(\frac{xc}{2} + \frac{xb}{2}, \frac{yc}{2} + \frac{yb}{2}\right)$

l) The slope of B′C′ = $\frac{xc-xb}{yc-yb}$

m) The slope of any line perpendicular to B′C′ then is $\frac{yb-yc}{xc-cb}$

The equation of the perpendicular bisector of B′C′ then is

$$n)\ y - \left(\frac{yc}{2} + \frac{yb}{2}\right) = \frac{yb-yc}{xc-xb}\left(x - \left(\frac{xb}{2} + \frac{xc}{2}\right)\right)$$

substituting the point A′(xa,ya) into this equation we get

$$o)\ ya - \left(\frac{yc}{2} + \frac{yb}{2}\right) = \frac{yb-yc}{xc-xb}\left(xa - \left(\frac{xb}{2} + \frac{xc}{2}\right)\right)$$

if this equation is true then D lies on the perpendicular bisector of EF, we simplify this equation and get

p) (2ya-yb-yc)(xc-xb) = (yb-yc)(2xa-xb-xc)

to verify equation p is a lot of work, it requires substituting the expressions that each of the variables in these equation are equal to (see equations e,f,g,h,i,j), and then carrying out the all arithmetic operations, and then verifying that the equation above is indeed true. If so we have proved that A′ lies on the perpendicular bisector of B′C′. This will not be done by hand, but will be performed using the computer program Derive. This computer verification has not been performed yet, it will be performed later. We will continue on with the proof as if this part of the proof has already been done.

The fact equation p is true is the final part of the proof that A′ lies on the perpendicular bisector of B′C′.

Now we desire to prove that B′ lies on the perpendicular bisector of A′C′. This is accomplished by noting that if the point we call B′ had been named A′, and if the point we now call C′ had been named B′, and if the point we now call A′ had been named C′, and if the point we now call A had been named C, and if B had been named A, and if C had been named B, the above proof would have proved that what we now call B′ lies on the perpendicular bisector of what we now call A′C′.

A similar proof will show that C′ lies on the perpendicular bisector of

A’B’.

Given that all vertices of the triangle A’B’C’ lie on the perpendicular bisector of the side opposite each vertex, triangle A’B’C’ is an equilateral triangle.*

Proof complete

* Here we will provide the outline of a proof showing if all vertices of a triangle lie on the perpendicular bisector of the side opposite the vertex, then the triangle is an equilateral triangle. ... Given triangle ABC, suppose A lies on the perpendicular bisector of BC, then AB=AC. If B lies on the perpendicular bisector of AC, then BA=BC. Substituting AB=AC into AB=BC, we have AC=BC. Therefore we have AB=AC=BC. Outline of proof complete.

^^

10.43)   Napoleon's Theorem 2

If equilateral triangles are placed on all three sides of any arbitrary triangle, such that the arbitrary triangle and each of the equilateral triangles share a common side, and the equilateral triangles are pointed inward with respect to the arbitrary triangle, prove that the centroids of these equilateral triangles are themselves vertices of an equilateral triangle. This equilateral triangle is referred to as the arbitrary triangle's, Napoleon triangle 2.

The proof of Napoleon’s Theorem 2 relies heavily upon the proof of Napoleon’s theorem 1. In Napoleon’s theorem 1, it was stated concerning the vertices of triangle ABC that A-B-C is counterclockwise around the triangle. If the word counterclockwise is changed to clockwise, then the proof changes, and proves Napoleon’s theorem 2, not Napoleon’s theorem 1. Proof complete:

Note: For a hint as to why this is to, focus on expression a) and the proof that follows

a) $$\frac{(A+B)}{2} + \&(B-A) * \frac{\text{sqrt}(3)}{6}$$

If A-B-C is counterclockwise, expression a) will be the point that is the centroid of the equilateral triangle with side AB, that is pointed outward from the triangle ABC. If A-B-C is clockwise, a) will be the point that is the centroid of the equilateral triangle with side AB, that is pointed inward to triangle ABC.

10.45) Beautiful Cable Support Structure

Two rings (circles) of radius r lie in parallel planes separated by a distance of h. One ring is directly on top of the other, such that the line connecting the centers of these rings is perpendicular to the planes containing the rings. Each ring has on it n (n= 3 or 4 or 5 or ...) evenly spaced hooks. Every hook on the bottom ring has directly over it, a hook on the top ring and every hook on the top ring has directly below it, a hook on the bottom ring. The bottom ring is suspended in the air by cables connecting it to the top ring in the following manner. Each hook of the top ring, connects to two hooks on the bottom ring as follows. A top ring hook connects not to the hook directly below it, but to a hook 1 or 2 or 3 ... hooks to the right AND to the left of the hook that is directly to below it. Question, what is the shape the outline of the cables connecting the two rings? When I was young, there was a community theater called the Valley Music Hall, that had such a setup. The lower ring supported a platform that had stage lights on it. Viewed from the side,(looking at the rings edge), the outline of the cables made a beautiful and interesting shape of what looked like a hyperbola or parabola pointing outward from and "rotating" around line connecting the centers of the top and bottom rings. The task here is to find the shape of the figure (its equation) being "rotated" around the center line. Is this rotated shape a parabola, or a hyperbola or something else?

Solution

Note: When using 3 dimensional coordinates, i.e. (1,-2,3), the first place is the x coordinate, the second place is the y coordinate and the third place is the z coordinate, which gives vertical position.

It is only to examine one cable given that every cable is similarly situated.

Our cable connects to the bottom ring at

a) (r,0,0)

Our cable connects to the top ring at

b) (r-d, sqrt(2rd-d^2),h)

We define the path of the cable using the parametric equation, in accordance with the given information.

(r,0,0) + t{(r-d,sqrt(2rd-d^2),h)-(r,0,0)}

which simplifies to

c) (r-dt,sqrt(2rd-d^2)*t,ht)

Letting f represent the unknown equation we are solving for, The distance from any point on the cable (x,y,z), to the line connecting the two rings is given by the equation,

d) $f(z)^2=x^2+y^2$;

Substituting c) into equation d) we get

e) $f(z)^2=(r-dt)^2+(2rd-d^2)*t^2$

which expands to and is the same equation as

$$\text{f) } f(ht)^2=\left(r-\frac{dht}{h}\right)^2+(2rd-d^2)*\left(\frac{ht}{h}\right)^2$$

From equation c) we see that

g) $z=ht$

substituting equation g) into equation f) we get

$$\text{h) } f(z)^2=\left(r-\frac{dz}{h}\right)^2+(2rd-d^2)*\left(\frac{z}{h}\right)^2$$

Equation h) is the equation of the unknown function we seek. We will continue on, inorder to determine the type of function that it is and to get it into a more suitable form.

When working with variable coefficients, its important to keep track of what are variables and what are “changeable constants”. Starting with equation h), z is the independent variable, z represents the vertical position of any point of our cable. f(z) is the dependent variable, and represents the distance of a point on the cable to the line connecting the centers of the rings.

We multiply both sides of h) by h^2, getting rid of the fractions.

i) $f(z)^2*h^2=(hr-dz)^2 + (2rd-d^2)*z^2$

We perform all multiplications and gather like terms of z together.

j) $f(z)^2*h^2=z^2(2rd)-z(2hrd)+(h^2*r^2)$

We factor out the term 2rd from the right side of this equation.

$$\text{k) } f(z)^2*h^2 = 2rd\left(z^2 - \frac{2hrd}{2rd} + \frac{h^2*r^2}{2rd}\right)$$

We cancel out like terms

```
                  (           h^2*r)
l) f(z)^2*h^2 = 2rd(z^2 – zh + -----)
                  (             2d )
```

We change the form of the right side of this equation by completing the square, distributing the 2rd to all terms inside the parenthesis, then simplifying.

```
                  (    h )2  (          rdh^2)
m) f(z)^2*h^2 = 2rd(z - --) + (r^2*h^2 - -----)
                  (    2 )   (           2   )
```

We divide both sides by h^2

```
            2rd(    h )2  (      rd)
n) f(z)^2 = ---(z - --) + (r^2 - --)   <----  final answer
            h^2(    2 )   (      2 )
```

Were f(z), on the left side of the equation not squared, equation n) would be the equation of a parabola in standard form. Because the square is there, this is an equation of a hyperbola. The hyperbola is a geometric shape studied in analytic geometry. f(z) is the distance from the rotated hyperbola to the segment connecting the centers of the rings (the pole). Notice this function is shifted by h/2. The center of the pole is the part of the pole closest to the hyperbola.

11.19a

Prove cos(2a)=cos^2(a)-sin^2(a) = 2cos^2(a)-1 = 1-2sin^2(a)

```
cos(2a)=
cos(a+a)=  ... making use of I14
cos(a)cos(a)-sin(a)sin(a)=
cos^2(a)-sin^2(a)=  ... making use of I1   <-----   I20
cos^2(a)-[1-cos^2(a)]=
cos^(a)-1+cos^2=
2cos^2(a)-1=   ... making use of I1   <-----  I20
2[1-sin^2(a)]-1=
2-2sin^2(a)-1=
1-2sin^2(a)    <-----  I20
```

therefore

cos(2a)=cos^2(a)-sin^2(a) = 2cos^2(a)-1 = 1-2sin^2(a)   <---- I20


```
11.19b

Prove sin(2a)=2sin(a)cos(a)

sin(2a)=
sin(a+a)=  ... making use of I16
sin(a)cos(a)+sin(a)cos(a)=
2sin(a)cos(a)

therefore

sin(2a)=2sin(a)cos(a)    <---- I21

--------------------------
--------------------------

11.19c

                   2tan(a)
Prove  tan(2a)=  ------------
                  1-tan^2(a)


tan(2a) =   ... making use of definition of tan(o)
sin(2a)/cos(2a) =   ... making use of I20 and I21
2sin(a)cos(a) / [cos^2(a)-sin^2(a)] =

   2sin(a)cos(a)
   -------------
      cos^2(a)
--------------------- =
  cos^2(a)-sin^2(a)
  -----------------
     cos^2(a)


2tan(a)
-----------
1-tan^2(a)

therefore

            2tan(a)
tan(2a)=  ------------   <---- I22
           1-tan^2(a)


--------------------------
--------------------------

11.20a
```

```
                  1+cos(2a)
Prove  cos^2(a)= ---------
                      2

I20  ->
cos(2a)=2cos^2(a)-1  ->
cos^2(a)= {1+cos(2a)}/2


therefore

           1+cos(2a)
cos^2(a)= ---------    <---- I23
               2

---------------------------
---------------------------

11.20b

                1-cos(2a)
Prove sin^2(a)= --------
                   2

I1 ->
sin^2(a)+cos^2(a)=1  ->
sin^2(a)=1-cos^2(a)  ->      substituting I23 into cos^2(a)
sin^2(a)=1-[{1+cos(2a)}/2}]  ->
sin^2(a)={1-cos(2a)}/2

therefore

               1-cos(2a)
sin^2(a)= 1 - -----------    <----- I24
                   2

---------------------------
---------------------------

11.20c

                 1-cos(2a)
Prove  tan^2(a)= ---------
                 1+cos(2a)


tan^2(a)=             by definition
{sin(a)/cos(a)}^2=
sin^2(a)/cos^2(a)=    applying I23 and I24
{1-cos(2a)/2}/{1+cos(2a)/2}=
```

```
{1-cos(2a)}/{1+cos(2a)}

therefore

           1-cos(2a)
tan^2(a)= -----------     <----- I25
           1+cos(2a)

---------------------------
---------------------------
```

11.21a

```
                        [1+cos(a)]
Prove cos(a/2)=(+/-)sqrt[--------]
                        [    2   ]
```

```
I23 ->
cos^2(a)={1+cos(2a)}/2  ->
cos(a)=(+/-)sqrt[{1+cos(2a)}/2]  ->  substituting a/2 into a we get
cos(a/2)=(+/-)sqrt[{1+cos(a)}/2]
```

therefore

```
                  [1+cos(a)]
cos(a/2)=(+/-)sqrt[--------]    <---- I26
                  [    2   ]

---------------------------
---------------------------
```

11.21b

```
                        [1-cos(a)]
Prove sin(a/2)=(+/-)sqrt[--------]
                        [    2   ]
```

```
I24 ->
sin^2(a)={1-cos(2a)}/2   ->
sin(a)=(+/-)sqrt[{1-cos(2a)}/2]    ->  substituting a/2 into a we get
sin(a/2)=(+/-)sqrt[{1-cos(a)}/2]
```

therefore

```
                  [1-cos(a)]
sin(a/2)=(+/-)sqrt[--------]    <---- I27
                  [    2   ]
```


11.21c

```
                        [1-cos(a)]
Prove tan(a/2)=(+/-)sqrt[--------]
                        [1+cos(a)]
```

```
I25 ->
tan^2(a)=(+/-)[{1-cos(2a)}/{1+cos(2a)}] ->
tan(a)=(+/-)sqrt[{1-cos(2a)}/{1+cos(2a)}]   ->  substituting a/2 into a
tan(a/2)=(+/-)sqrt[{1-cos(a)}/{1+cos(a)}]
```

therefore

```
                  [1-cos(a)]
tan(a/2)=(+/-)sqrt[--------]    <---- I28
                  [1+cos(a)]
```

13.4) Preliminary Exercises to Prepare For The Following Problems

Study the line forms above and then without looking, a) write the name of each line form, then write an equation of a line in this form, using numbers as coefficients. b) write the name of each line form then write an equation of a line in this form, using variables as coefficients as shown above. Where applicable, write what information a line in this form tells you about the line. For example, write "slope is m, passes through point (x1,y1)". c) Given a line equation in general form, 3x+2y-8=0, what is the point on this line where x = 10? (that has an x coordinate of 10)? d) Make use of the technique of the last problem "c" to determine the slope of this line. e) Can you think of a second method to determine the slope of this line? f) Calculate the slope of this line using this second method also.

a) -- A set of answers is given below, your answers may differ.

```
slope .. y=3x
slope intercept .. y=2x+5
point slope .. y-7=4(x+2)

                   8-2
two point .. y-8= -----(x-3);
                   3+7

standard form .. -2x+12y=10
```

general form .. 4x-3y+5=0

b)
slope .. y=mx .. slope is m, passes through origin
slope intercept .. y=mx+b .. slope is m, crosses y axis at b
point slope .. y-y1=m(x-x1) .. slope is m, passes through point (x1,y1)

```
                       x   y
intercept intercept .. - + - = 1 .. intersects: x axis at a, y axis at b
                       a   b
```

```
                y2-y1
two point  y-y1=-----(x-x1) - passes through points (x1,y1) and (x2,y2)
                x2-x1
```

standard form .. ax+by=c
general form .. ax+by+c=0

c) Given the line 3x+2y-8=0, using equation manipulation we solve for y, y=(8-3x)/2, we then substitute 10 in for x getting y=(8-3*10)/2 = -11. Therefore where x=10, y=-11. The point we are looking for is (10,-11).

d) From the last problem we know the line 3x+2y-8=0 contains the point (10,-11). We use the same method to find another point on this line, we ask the question what is the point on this line where x=6. From the last problem we know y=(8-3x)/2, substituting 6 into this equation we get y={8-3(6)}/2 = -5. Therefore we know the point (6,-5) is also on this line. Therefore we have the points P1(10,-11) and P2(6,-5) as being on the line. The slope of the line is therefore m= (y2-y1)/(x2-x1) = {-5-(-11)}/(6-10) = (-5+11)/(6-10) = 6/(-4) = -3/2 so slope of line = -3/2.

e) To find the slope of 3x+2y-8=0 using a different method, we can use equation manipulation to change the form of this line from general form to slope intercept form. Once we have the slope intercept form, we can easily see the slope of the line. We do this in "f" below.

f) 3x+2y-8=0 -> 2y=8-3x -> y=(8-3x)/2 -> y=4-(3/2)x -> y= -(3/2)x+4. Now that the line is in this form, we see the slope of the line is -3/2. Which is the same answer as before.

13.5) For the line 3x-7y=5; a) Where the x coordinate is 11, what is the y coordinate? b) Where the y coordinate is -2, what is the x coordinate? c) Determine if the following points are on the line, (4,1),(-2,3).

a) Given the line 3x-7y=5, we solve for y. 3x-7y=5 -> -7y=5-3x -> y=(5-3x)/(-7). We substitute 11 in for x into this equation getting, y={5-3(11)}/(-7) -> y= 28/7.

b) Given the line 3x-7y=5, we solve for x. 3x-7y=5 -> 3x=7y+5 -> x=(7y+5)/3. Substituting -2 for y into this equation we get x={7*(-2)+5}/3 -> x=-3.

c)

Given the point (4,1) we substitute x=4 and y=1 into the line equation 3x-7y=5 -> 3(4)-7(1)=5 -> 12-7=5 -> 5=5. This is a true statement, therefore the point (4,1) is on line 3x-7y=5.

Given the point (-2,3) we substitute x=-2 and y=3 into the line equation 3x-7y=5 -> 3(-2)-7(3)=5 -> -6-21=5 -> -27=5. This is a false statement, therefore the point (-2,3) is not on line 3x-7y=5.

13.6) Slope Intercept -- y=2x-3 --

a) By inspection, give the slope and the y intercept of this line. Verify mathematically that your answer is correct, by verifying that the proposed y intercept is indeed a point of this line, then determine a total of two points that are on this line then use these two points to calculate the slope of this line.
b) Graph this line.
c) Find the point of this line that has an x coordinate of 5. Make use of this point to convert the equation into point slope form. Check your answer by converting the point slope form of the line, back to the slope intercept form using equation manipulation.
d) Calculate the x and y intercept of this line, make use of this information to convert this equation into intercept intercept form. Check your answer by converting the slope intercept equation of this line into intercept intercept form once again, this time by means of equation manipulation.
e) Convert this equation into two point form by finding and making use of any two points of your choosing except a point on either axis). Check your answer by manipulating the two point form of this line back into slope intercept form using equation manipulation.
f) Convert this equation into standard form via equation manipulation.
g) Convert this equation into general form via equation manipulation

a) By inspection it is seen that the line y=2x-3 has a slope of 2 and a y intercept of -3. Verifying this answer, first we find two points that are on this line and from these two points, we calculate the slope of the line. From y=2x-3, we see where x=0 that y=-3, and where x=1, y is -1. Therefore the points P1(0,-3) and P2(1,-1) are on this line. (Where slope is designated as m) we have m=(y2-y1)/(x2-x1) or m={-1-(-3)}/(1-0) -> m=2/1 -> m=2. This verifies the slope of this line is 2. To verify that the y intercept of -3, we need to verify that the point (0,-3) is on this line. To do this we substitute x=0 and y=-3 into y=2x-3 and see if it leads to a true equation. y=2x-3 -> -3=2(0)-3 -> -3=-3. This equation is true, meaning the point (0,-3) is on the line y=2x-3, meaning the y intercept of the line y=2x-3 is -3.

b) To graph this line, use a piece of graph paper. Find two points that are on this graph, and then draw a line through these two points.

c) To find the point of y=2x-3 that has an x coordinate of 5, we substitute 5 in for x into this equation and solve for y, getting y=2(5)-3 -> y=7. Therefore the point of this line that has an x coordinate of 5 is (5,7). From part "a" we know this line has a slope of 2. Using this information, we know a point slope form of this line is y-7=2(x-5), using equation manipulation we convert this line back into point slope form as follows y-7=2(x-5) -> y=2(x-5)+7 -> y=2x-10+7 -> y=2x-3. This is the same equation we started with, so we did this problem correctly.

d) To find the x intercept of this line, we substitute 0 into y in this equation, getting the following. y=2x-3 -> 0=2x-3 -> 3=2x -> x=3/2. To find the y intercept of this line, we substitute 0 into x in this equation, getting the following. y=2x-3 -> y=2(0)-3 -> y=-3. Using this information we know the intercept intercept form of this line is x/(3/2)+y/(-3)=1. Once again we convert y=2x-3 into intercept intercept form as follows. y=2x-3 -> -2x+y=-3 [we divide both sides of this equation by -3] -> (-2/-3)x + y/(-3)=1 -> (2/3)x + y/(-3)=1 -> x/(3/2) + y/(-3)=1. Getting the same answer both times we have verified our answer.

e) On the line y=2x-3, where x is 1, y is y=2(1)-3 -> y=2-3 -> y=-1. Therefore the point (1,-1) is a point of this line. On this line where x=2, y is y=2(2)-3 -> y=1. Therefore the point (2,1) is also on this line. So the points P1(1,-1) and P2(2,1) are on this line. Therefore a two point form of this line is

$$y-1= \frac{1-(-1)}{2-1}(x-2)$$, we manipulate this back into slope intercept form, ->

$$y-1= \frac{2}{1}(x-2) \rightarrow y-1=2(x-2) \rightarrow y-1=2x-4 \rightarrow y=2x-3$$

f) y=2x-3 -> 2x-y-3=0 -> 2x-y=3

g) y=2x-3 -> 2x-y-3=0

14.4) Lines Intersection
Find the point of intersection of each of the following 2 line pairs in each of the following three ways. I) Graph the lines, find the points where they intersect graphically. II) Equation addition (commonly called elimination). III) Substitution. Compare answers. a*) 3x-2y=8; 7x+3y=5. b) line has a slope of 2 and a y intercept of 5; line passes through point (-1,2) and has a slope a –7.

I) This solution will not be given here.

```
IIa) 3x-2y=8   -> 3(3x-2y=8)  ->  9x-6=24   -> adding these two
     7x+3y=5      2(7x+3y=5)     14x+6y=10    equations together gives
```

23x=34 -> x=34/23. Taking one of the original given equations 7x+3y=5 and solving for y we have 7x+3y=5 -> 3y=5-7x -> y=(5-7x)/3. Next we substitute the value of x into this equation getting y={5-7(34/23)}/3 -> y= -41/23. so x=34/23 and y=-41/23.

IIb) This solution will not be given here.

```
IIIa) 3x-2y=8 ->  7(3x-2y=8) -> 21x-14y=56 -> 21x=56+14y ->
      7x+3y=5     3(7x+3y=5)    21x+ 9y=15    21x=15- 9y
```

56+14y=15-9y -> 23y=-41 -> y= -41/23. Taking one of the original given equations 3x-2y=8 we solve for x, 3x-2y=8 -> 3x=8+2y -> x=(8+2y)/3 next we substitute the value for y into this last equation leading to x={8+2(-41/23)}/3 -> x= 34/24 so x= 34/24 and y= -41/23

IIIb) This solution will not be given here

14.5) Recommended Problem: Given the line equations y=mx+a and y=nx+b. Once x is solved for, y can be solved for by substituting the value of x into either of these equations. Prove that it doesn't matter which one.

y=mx+a -> mx+a=nx+b -> mx-nx=b-a -> x(m-n)=b-a -> $x = \frac{b-a}{m-n}$

substituting the value of x into y=mx+a gives $y=m\left(\frac{b-a}{m-n}\right)+a$

substituting the value of x into y=nx+b $y=n\left(\frac{b-a}{m-n}\right)+b$

so we have to prove prove that $m\left(\frac{b-a}{m-n}\right)+a = n\left(\frac{b-a}{m-n}\right)+b$

$$m\left(\frac{b-a}{m-n}\right)+a = n\left(\frac{b-a}{m-n}\right)+b \;\rightarrow$$

$$m(b-a)+a(m-n) = n(b-a)+b(m-n) \;\rightarrow$$

$$mb-ma + am-an = nb-na + bm-bn \;\rightarrow$$

$$-an = -na$$

This is a proof of what we set out to prove, but to make this proof more plain, we take this proof and make a new proof by working backwards which leads to the proof below.

$-an = -an \rightarrow$ (adding ma-ma+mb to the left side and (nb-nb+mb to the right side gives)

$$-an + ma-ma+mb = -an + nb-nb+mb \;\rightarrow$$

$m(b-a)+a(m-n) = n(b-a)+b(m-n) \rightarrow$ (dividing both sides by m-n gives)

$m\left(\frac{b-a}{m-n}\right)+a = n\left(\frac{b-a}{m-n}\right)+b$ <- this is what we set out to prove

Proof Complete

16.10) a) Using method 2, find the zeros of $Ax^2+Bx+C=0$. Hint: If you find this difficult, do this problem first using numbers as coefficients, then use this as a guide to do the given problem.
b) Most likely, the expression you derived in part a was not in the same form as the expression given below. Using algebraic manipulation, put the expression you derived in part a into the following form.

$\frac{-B \pm \sqrt{B^2-4AC}}{2A}$ <- this expression is referred to as the quadratic formula.

$$Ax^2+Bx+C=0 \;\rightarrow$$

$$x^2 + \frac{B}{A}x + \frac{C}{A} = 0 \;\rightarrow$$

```
     (B)    (B )2   C   (B )2
x^2 +(-)x + (--)  + - = (--)  ->
     (A)    (2A)    A   (2A)


[  (B )]2   C   (B )2
[x+(--)]  + - = (--)   ->
[  (2A)]    A   (2A)


[  (B )]2   (B )2   C
[x+(--)]  = (--)  - -  ->
[  (2A)]    (2A)    A


  (B )              [ (B )2   C ]
x+(--)  = +/- sqrt[ (--)  - - ]  ->
  (2A)              [ (2A)    A ]



  (B )              [ (B )2   4AC  ]
x+(--)  = +/- sqrt[ (--)  - ---- ]  ->
  (2A)              [ (2A)    4A^2 ]


   -B           [ (B )2   4AC  ]
x= --- +/- sqrt[ (--)  - ---- ]  ->
   2A           [ (2A)    4A^2 ]



x= -B +/- sqrt(B^2-4AC)
   --------------------   <--- This is the Quadratic Formula
           2A
```

17.13) a) Shifting y-5=-2(x+3)^2, 1 to the right and up 2 produces the same result as adding what function to this parabola? b) Verify that the answer you got in "a" is correct. c) Prove that adding a non-vertical line to any non-rotated parabola results in a parabola, that is congruent to the original parabola but shifted to a new location. d) Prove that any non-rotated parabola may be shifted to any desired position by adding to it, an appropriate line.

When y-5=-2(x+3) is shifted 1 to the right and 2 upward, according to the shifting theorem, this parabola becomes

(y-2)-5=-2[(x-1)+3]^2 = y-7=-2(x+2)^2

We let ? be the function that if added to the function y-5=-2(x+3)^2 the sum of these two function will be the same as shifting y-5=-2(x+3)^2 one to the right and 2 upward.

the function y-5=-2(x+3)^2 + the function ? = the function y-7=-2(x+2)^2

implies

the function y=-2(x+3)^2+5 + the function ? = the function y=-2(x+2)^2+7

implies

-2(x+3)^2+5 + ? = -2(x+2)^2+7 ->

? = -2(x+2)^2+7 - [-2(x+3)^2+5] =

-2(x+2)^2+7 + 2(x+3)^2-5 =

-2(x^2+4x+4)+7 + 2(x^2+6x+9)-5 =
-2x^2-8x-8+7 + 2x^2+12x+18-5 =

4x+12

so y=4x+12 is the function we seek.

Shifting y-5=-2(x+3)^2, 1 to the right and up 2 produces the same result as adding what function to this parabola?

b) Verify that the answer you got in "a" is correct.

From A we have

the function y-5=-2(x+3)^2 + the function ? =
the function y-5=-2(x+3)^2 shifted 1 right, 2 up

implies

the function y-5=-2(x+3)^2 + the function 4x+12 =
the function (y-2)-5=-2{(x-1)+3}^2

implies

the function y-5=-2(x+3)^2 + the function 4x+12 =
the function y-7=-2{x+2}^2

implies

the function y-5=-2(x+3)^2 + the function 4x+12 =
the function y-7=-2{x+2}^2

implies

the function y=-2(x+3)^2+5 + the function 4x+12 =
the function y=-2{x+2}^2+7

implies

the function y=-2(x^2+6x+9)^2+5 + the function 4x+12 =
the function y=-2{x+4x+4}^2+7

implies

the function y=-2x^2-12x-18 +5 + the function 4x+12 =
the function y=-2x-8x-8 +7

implies

-2x^2-12x-18 +5 + 4x+12 = -2x-8x-8 +7

implies

-12x-18 +5 + 4x+12 = -8x-8 +7

-8x-1=-8x-1

This last statement being true, implies what we set out to prove is true

proof complete

----

c) Prove that adding a non-vertical line to any non-rotated parabola results in a parabola, that is congruent to the original parabola but shifted to a new location.

Any non rotated parabola + a general non vertical line equals

The function y-y0=a(x-x0)^2 + the function y=mx+b equals

The function y=a(x-x0)^2+y0 + the function y=mx+b equals

a{x^2-(2x0)x+x0^2}+y0 + mx+b equals

ax^2-(2ax0)x+ax0^2+y0 + mx+b equals

```
  ax^2+(m-2ax0)x+(ax0^2+y0+b) equals

                (m-2ax0)2   (m-2ax0)2
ax^2+(m-2ax0)x+ (------)  - (------)  + (ax0^2+y0+b) equals
                (  2   )    (  2   )


(        m-2ax0 )2   (m-2ax0)2               )
( ax^2 + ------ )  - (------) + (ax0^2+y0+b) )  equals the function
(          2    )    (  2   )                )


    [        m-2ax0 )2   {               (m-2ax0)2} ]
y = [ ax^2 + ------ )  + {(ax0^2+y0+b) - (------) } ]
    [          2    )    {               (  2   ) } ]
```

This last equation is has the form of a general form of a parabola, proving that

Any non rotated parabola + a general non vertical line is a parabola.

Proof Complete

----

d) Prove that any non-rotated parabola may be shifted to any desired position by adding to it, an appropriate line.

any non-rotated parabola + appropriate line if it exists = the non-rotated parabola after being shifted anywhere

expressed symbolically as

```
the function (y-y0)=c(x-x0)^2 + ? = the function (y-b)=c(x-a)^2 ->

? = the function (y-b)=c(x-a)^2 - the function (y-y0)=c(x-x0)^2 ->

? = the function y=c(x-a)^2+b - the function y=c(x-x0)^2+y0  ->

? = c{x^2-2ax+a^2)+b - [ c{x^2-(2x0)x+x0^2}+y0 ] ->

? = cx^2-2cax+ca^2+b - [ cx^2-(2cx0)x+cx0^2+y0 ] ->

? = cx^2-2cax+ca^2+b - cx^2+(2cx0)x-cx0^2-y0  ->

? = cx^2-cx^2  +(2cx0)x-2cax  +ca^2+b-cx0^2-y0  ->

? = 2c(1-a)x+(ca^2+b-cx0^2-y0)
```

so the appropriate line we seek does exist and it is in slope intercept form

```
y = 2c(1-a)x+(ca^2+b-cx0^2-y0)
```

Proof Complete

Notes:

Define domain and range

put compression proof and shifting proof in at the beginning as two of the coordinate geometry postulates

ensure general point and general line are defined early enough.

Move Vectors to problem section

Derivatives: Verify maximums / minimums via shifting theorem

Eric Widdison: Look at and impliment his suggestions.

In problems where it says, do problem 2 ways, .. provide solution.

Algebra Section preface text
section also covers polynomials, the quadratic formula and its derivation. The quadratic formula is sometimes useful in solving geometry problems, in addition deriving it gives students experience doing algebra proofs.

Put this where it belongs
There are several example problems given in this section. In addition there are several solutions for problems given in the in the back of the book. Not all problems have solutions in the back of the book. Problems that do not will be numbered (for example) as 1), 4), 11) etc. Problems that do have solutions in the back of the book will be numbered (for example) as 12.1), 12.4), 12.11). Please read the Preface in the front of the book to learn more about problems which have solutions in the back of the book and how to easily access these problems.

Provide proofs and problems: Parallel Line theorems easier.

Provide proofs and problems: Perpendicular theorems involving undefined.